İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

STRESS ANALYSIS ON TENDON STRUCTURE

M.Sc. Thesis by

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Department : **Mechanical Engineering**

Programme: **Solid Mechanics**

Supervisor : Prof. Dr. Tuncer TOPRAK

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Levent KIRKAYAK

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SYMBOL LIST

u, x, v, w ..	: Vector
I	: Identity tensor
O	: Zero tensor
F	: Second order tensor
W	: Skew-symmetric tensor
τ	: Kirchhoff stress tensor
C	: Cauchy-Green tensor
E	: Euler-almansi strain tensor
P	: The first piola-kirchoff stress tensor
S	: The second piola-kirchoff stress tensor
L	: Velocity gradient tensor
D	: Strain rate tensor
σ	: Cauchy stress tensor
Q	: Orthogonal tensor
Ψ	: Strain energy function
B	: Left cauchy green tensor
n	: Outward normal
T	: Cauchy traction vector
i,*j*	: Index parameters
x	: Cross product
$\otimes$	: Tensor product
$T^{(n)}$	: Traction vector
β_0	: Boundary surface
J	: Volume ratio
ω	: Current domain
Ω_o	: Reference domain
ε	: Internal energy density
∇_o	: The referential del operator
δ	: Kronocker delta
q_0	: The referential heat flux vector
D_{int}	: Inner energy
p	: Intermediate lagrange multiplier
λ	: Stretch ratio
μ_i, α_i	: Ogden material parameter
G	: Neo-hookean material coefficient
C_{10}, C_{01}....	: Material model coefficients.

SUMMARY

Biomechanics is the application of mechanics principles to living organisms. This area comes from realization that biology can no more be understood without the underlying principles of mechanics that drive the system. For an organism, biomechanics help us to understand its normal function, predict changes due to alterations, and propose methods of artificial intervention. Thus diagnosis, surgery, and prosthesis are closely associated with biomechanics.

The soft biological tissues (skin, tendon, and ligament) play an important role in the mechanical integrity of the body. Like the other soft tissues, tendons exhibit nonlinear behavior even under sub-physiologic loading, which is difficult to analyze

In this study, a series of experiments were conducted to obtain the mechanical properties of tendon structures. The experimental data has been fit to various nonlinear material models A comparison between these models have been discussed enlightening the pros and cons of each different formulation, and postulates over future studies have been discussed.

ÖZET

Biyomekanik mekaniğin temel prensiplerinin canlı organizmalara uygulanmasıdır. Biyomekanik, organizmaların normal fonksiyonlarını nasıl yerine getirdiğinin ve değişikliklere karşı nasıl cevap verdiğinin anlaşılmasını sağlar ve bunların suni olarak yapılması için methodlar önerir.Bu nedenle cerrahi ve protez bilimi biyomekanikle yakından ilişkilidir.

Deri, tendon ligament gibi yumuşak dukular vücudun mekanik bütünlüğünde önemli rol oynarlar.Diğer yumuşak dokularda olduğu gibi, tendon da yükleme altında doğrulsal olmayan davranış gösterir ve bu analiz edeilmesini zorlaştırır.

Bu çalışmada tendon yapısının mekanik özelliklerinin belirlenmesi için bir takım deneyler yapılmıştır.Deney sonucu elde edilen veriler doğrusal olmayan olmayan çeşitli malzeme modellerine uygulanmıştır.Bu modellerin pozitif ve negatif yönleri arasında karşılaştırma yapılmış ve ileriye dönük nasıl geliştirileceği tartışılmıştır.

1 INTRODUCTION

The soft biological tissues (skin, tendon, and ligament) play an important role in the mechanical integrity of the body. Soft tissues have the following functions: to protect the body for the skin, to transfer loads between bones for the ligaments, or between muscles and bones for the tendons. Soft tissues may be distinguished from other body tissues (like bones) with their flexibility and their relatively soft mechanical properties. The role of tendons is to transmit different body forces to the bones, whereas that of ligaments is to handle the stability of joints and restrict their ranges of motion [1].

The soft biological tissues are mainly made of collagen and elastin proteins, which bring special mechanical properties. Many of the tissues can be stretched 15% without damage. They also have an important viscous component in their behaviors [1].

The properties and the mechanical behavior of these tissues have been of great interest to many researchers. The mechanical properties of these materials must be provided in the form of a stress-strain constitutive relationship. Like the other soft tissues, tendons exhibit both nonlinear and viscoelastic behavior under loading, which is more difficult to analyze [1].

In this study, a series of experiments were conducted to obtain the mechanical properties of tendon structures. The experimental data has been fit to various nonlinear material models. A comparison between these models have been discussed enlightening the pros and cons of each different formulation, and postulates over future studies have been discussed.

1.1 The Biological Structure of Tendon

The schematic view of tendon structure is given in the Figure 1.1. The largest structure in figure is tendon or the ligament .A ligament or tendon then is split into smaller entities that called fascicles. A fascicle contains basic fibrils of ligaments or tendons, and fibroblasts, which are the biological cells that produce the ligament or tendon. There is a structural characteristic at this level that plays a significant role in the mechanics of ligaments and tendons: the crimp of the fibril. The crimp is the waviness of the fibril; we will see that this contributes significantly to the nonlinear stress strain relationship for ligaments and tendons and indeed for basically all soft collagenous tissues [1].

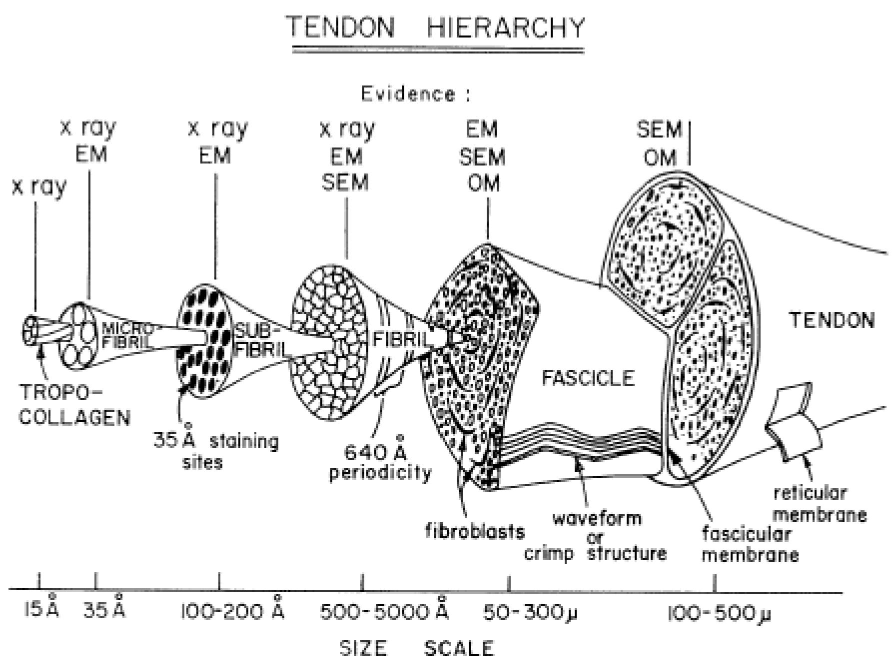


Figure 1.1: The fibrous structure of tendon

1.2 Tendons

The basic anatomic properties of tendon is:

- Tendons contain collagen fibrils (Type I)
- Tendons contain a proteoglycan matrix
- Tendons contain fibroblasts (biological cells) that are arranged in parallel rows

Basic Functions;

- Tendons carry tensile forces from muscle to bone
- They carry compressive forces when wrapped around bone like a pulley

The properties of type I Collagen:

- ~86% of tendon dry weight
- Glycine (~33%)
- Proline (~15%)
- Hydroxyproline (~15%, almost unique to collagen, often used to identify)

Blood Supply:

- Vessels in perimysium (covering of tendon)
- Periosteal insertion
- Surrounding tissues

1.3 Ligaments

Anatomy :

- Similar to tendon in hierarchical structure
- Collagen fibrils are slightly less in volume fraction and organization than tendon
- Higher percentage of proteoglycan matrix than tendon
- Fibroblasts

Blood Supply :

- Microvascularity from insetion sites
- Nutrition for cell population; necessary for matrix synthesis and repair [1].

1.4 General overview of ligament and tendon mechanics

As with all biological tissues, the hierarchical structure of ligaments and tendons has a signficant influence on their mechanical behavior. Unlike bone, however, not nearly as much quantiative structure function relationships, either experiment/statistical or analytical, have been derived for ligaments and tendons. This is for two reasons. The first reason is the hierarchical structure of ligaments and tendons is much more difficult to quantify than bone. And the second, ligaments and tendons exhibit both nonlinear and viscoelastic behavior even under physiologic loading, which is more difficult to analyze than the linear behavior of bone [6].

1.5 Mechanical Properties of Tendon

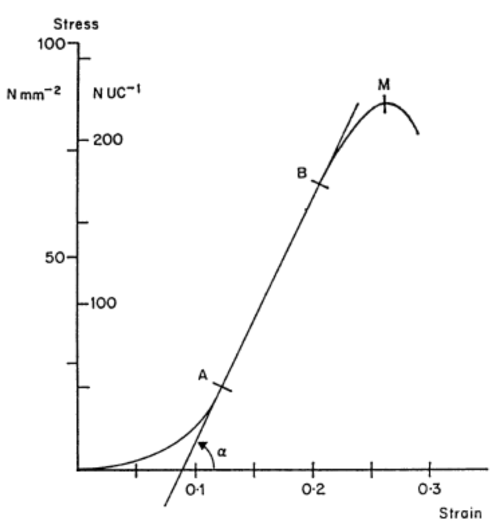
Soft tissues exhibit clearly quasi-incompressible, non-homogeneous, non-isotropic, non-linear viscoelastic materials in large deformation mechanical properties of soft tissues are due to their structure rather than to the relative amount of their constituents (Fung ,1987). The tendon fascicles are organized in hierarchical bundles of fibers arranged in a more or less parallel fashion in the direction of the effort handled. A close look at the fiber networks shows that this parallel arrangement is more irregular and distributed in more directions for ligaments than for tendons [6].

The general procedure for defining a mechanical model for tendons consists of describing the evolution of a (hypothetical) continuous medium using the continuum mechanics theory, approximating its geometry to a set of discrete finite elements if required, and simulating its evolution using incremental/iterative procedures. In this approach, the mechanical properties of the material must be provided in the form of a stress-strain constitutive relationship.

Various such biomechanical relationships have been proposed for soft tissue modeling.The main property of soft tissues may be outlined as being their nonlinear elasticity. Kwan described the phenomenon as follows: "Under uniaxial tension,parallel-fibered collagenous tissues exhibit a non-linear stress-strain relationship characterized by an initial low modulus region, an intermediate region of gradually increasing modulus, a region of maximum modulus which remains relatively constant, and a final region of decreasing modulus before complete tissue rupture occurs. The low modulus region is attributed to the removal of the undulations of collagen fibrils that normally exist in a relaxed tissue. As the fibrils start to resist the tensile load, the modulus of the tissueincreases. When all the fibrils become taut and loaded, the tissue modulus reaches a maximum value, and thereafter, the tensile stress increases linearly with increasing strain. With further loading, groups of fibrils begin to fail, causing the decrease in modulus until complete tissue rupture occurs." A typical tensile curve is shown in Fig. 1.2. From a functional point of view, the first parts of the curve are more useful since they correspond to the physiological range in which the tissue normally functions [10].

The before mantioned experiment reveals the relationship between stress and strain in the static case. However, when the equilibrium is not reached, a history-dependent component exists in the mechanical behavior of living tissues [6].

When measured in dynamic extension, the stress values appear higher than those at equilibrium, for the same strain. The resulting tensile curve appears steeper than the one at equilibrium (Fig. 1.3). When a tissue is suddenly extended and maintained at its new length, the stress gradually decreases slowly against time. This phenomenon is called stress relaxation (Fig 1.4a). When the tissue is suddenly submitted to a constant tension, its lengthening velocity decreases against time until equilibrium. This phenomenon is called creep (Fig. 1.4b). Under cyclic loading, the stress strain curve shows two distinct paths corresponding to the loading and unloading trajectories. This phenomenon is named hysteresis (Fig. 1.4c). As a global statement, the stress at any instant of time depends not only on the strain at that time, but also on the history of the deformation. These mechanical properties, observed for all living tissues, are common features of a physical phenomenon named viscoelasticity [6].



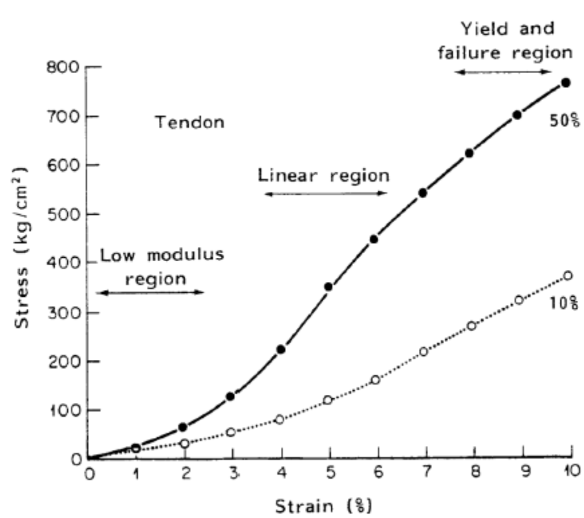
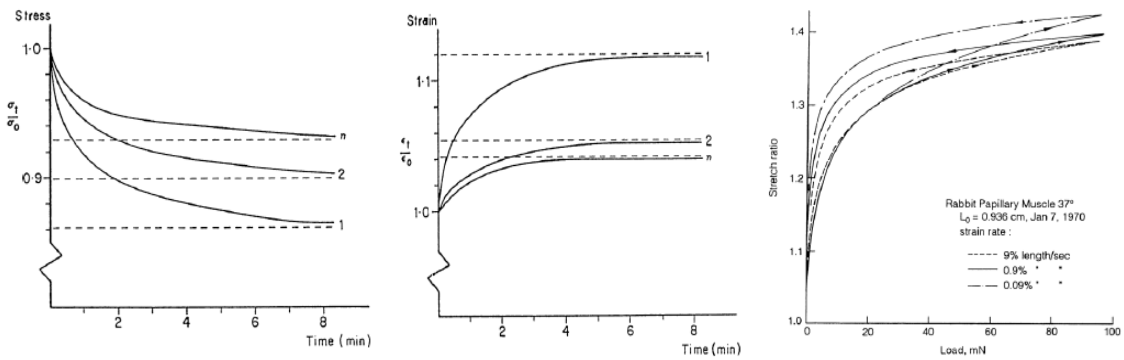


Figure 1.2: a) Load-extension curve b) Influence of the train rate



a) stress-relaxation [12]. b) creep [12]. c) hysteresis [12]

Figure 1.3: Viscoelastic behaviors

The compressibility of soft tissues has been investigated very little. Soft tissues are however usually assumed to be incompressible materials [13]. Besides, when loading-unloading cycles are applied on the tissue successively up to the same stress level, the stressstrain curve is gradually shifted to the right. After a number of such cycles, the mechanical response of the tissue enters a stationary phase and the results become reproducible from one cycle to the next. This phenomenon is due to the changes occurring in the internal structure of the tissue, until a steady state of cycling is reached. This initial phase of behavior common to all living tissues is usually used as preconditioning of the tissues prior to experimentation (Fig. 1.4) [14].

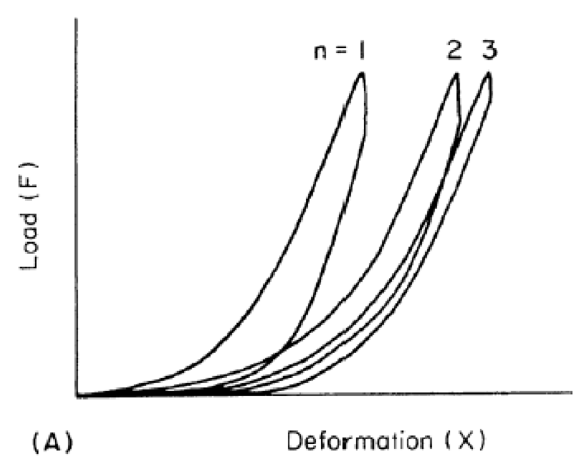


Figure 1.4: Preconditioning

A further property may be outlined as the propensity to undergo large deformations. In normal tensile tests, graphs are plotted for the Lagrangian stress T with respect to the Lagrangian strain ε. Sometimes, this Lagrangian stress is taken for the true stress (Cauchy stress) σ in the resulting constitutive equation. It must be emphasized that this substitution is valid only for strains smaller than 2% of the resting length. However, soft tissues are likely to exceed this limit in their physiological range of functioning, so that, in most cases, this assumption no longer applies [1].

For example, a common strain for tendons is around 4 % but they may extend up to 10 % of their original length. One then talks about finite strain, or large deformation. Summarizing, soft tissues may be characterized as quasi-incompressible, non-homogeneous, non-isotropic, non-linear viscoelastic materials likely to undergo large deformations. Though with different proportions depending on the tissue, these properties may be attributed to all soft-tissues, passive muscle included. Numerous investigations have been lead towards the constitutive modeling of these materials [1].

2 CONTINUUM MECHANICS APPROACH

2.1 Basic algebra of vectors and tensors

2.1.1 Direct Notation

A vector is a mathematical quantity possessing characteristics of magnitude and direction. For this reason, vectors are often represented by arrows, the length of which denotes the magnitude. In other words, a vector designated by **u**, **v**, **w** … is a directed line element in space. It is a model for physical quantities having both direction and length, for example, force, velocity or acceleration. The two vectors that have the same direction and length are said to be equal [4].

The sum of vectors yields a new vector, based on the parallelogram law of addition. The following properties,

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}\,, \tag{2.1}$$

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}), \tag{2.2}$$

$$\mathbf{u} + \mathbf{0} = \mathbf{u}\,, \tag{2.3}$$

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0} \tag{2.4}$$

Hold, where “0” denotes the unique zero vector with unspecified direction and zero length [3].

Besides addition and subtraction, which can be accomplished using the parallelogram law with the arrow representation, three “vector operations” of utmost importance are the scalar (or, dot) product,

$$\mathbf{u}.\mathbf{v} = \mathbf{a} \text{ where } a = |u|.|v|.Cos\theta \tag{2.5}$$

The vector (or, cross) product,

$$u \times v = w \text{ where } w = |u|.|v|.Sin(\theta)e \tag{2.6}$$

And the tensor (or, dyadic) product,

$$u \otimes v = T \tag{2.7}$$

Herein, θ is the angle between vectors u and v, $|...|$ denotes the magnitude of a vector, e is a unit vector (i.e., $|e| = 1$) perpendicular to the plane containing u and v, T is a second-order tensor. The magnitude of the vector w is found by $|w| = (w.w)^{\frac{1}{2}}$, and a unit vector e in the direction of w can be found via $e = \frac{w}{|w|}$. Two vectors, u and v are aid to be orthogonal if

$$u.v = 0. \tag{2.8}$$

Collectively these equations above reveal that two vectors can "operate" on one another to yield a scalar, anew vector, or a second order tensor. Higher order tensors, as, for example, the third order tensor $u \otimes v \otimes w$, are equally easy to obtain. [4]

Recall that the dot product commutes, that is

$$u.v = v.u \tag{2.9}$$

In contrast,

$$u \times v = -v \times u \tag{2.10}$$

and in general,

$$u \otimes v \neq v \otimes u \tag{2.11}$$

Also note that,

$$(w \otimes u).v = w.(u.v) = w.a \tag{2.12}$$

$$w.(u \otimes v) = (w.u).v = b.v \tag{2.13}$$

which shows, for example, that a “dot product” between a second-order tensor $w \otimes u$ and a vector v, yields a vector in the direction of w that has a different magnitude. Moreover, the “.” operation takes precedence over the “$\otimes$”; thus the parenthesis can be deleted. The last two equations reveal, therefore, that a second order tensor transforms a vector in to a new vector, which is why tensors are called linear transformations. Many of the basic operations for second order tensors, say **S** and **T**, are similar to those for vectors. For example, recall the basic associative and distributive laws for vectors,

$$(a.u).v = a.(u.v) = u.(a.v) \tag{2.14}$$

and,

$$(u + v).w = u.w + v.w\,. \tag{2.15}$$

These laws are similar for second order tensors, thus

$$(a.\mathbf{S}).v = a.(\mathbf{S}.v) = \mathbf{S}.(a.v) \tag{2.16}$$

and,

$$(\mathbf{S} + \mathbf{T})v = \mathbf{S}.v + \mathbf{T}.v \tag{2.17}$$

Satisfaction of these two equations ensures that the set of all second order tensors from a vector space. Likewise,

$$(au + bv) \otimes w = a(u \otimes w) + b(v \otimes w) \tag{2.18}$$

Additional operations important for second order tensors include the transpose $(...)^T$, trace(…) and determinant det(…). In particular,

$$(u \otimes v)^T = (v \otimes u) \tag{2.19}$$

which is to say that the transpose interchanges the order of the vectors that constitute the dyad;

$$tr(u \otimes v) = u.v \tag{2.20}$$

Thus the trace of a tensor yields the scalar product of the vectors constituting the dyad; and

$$\det \mathbf{T} = \det[\mathbf{T}] \tag{2.21}$$

Where (…) denotes a matrix representation of **T.** The determinant of a tensor thereby yields a scalar, one that equals the determinant of the matrix of components of the tensor. Another scalar measure of a second order tensor is its magnitude, given as

$$|\mathbf{T}| = \sqrt{tr\left(\mathbf{T}.\mathbf{T}^T\right)} \tag{2.22}$$

A second order tensor, say $w \otimes u$, can also act on another second order tensor, say $v \otimes x$, to yield a second order tensor, viz.;

$$w \otimes u.v \otimes x = (u.v)w \otimes x = a(w \otimes x) \tag{2.23}$$

or either of two scalars,

$$w \otimes u : v \otimes x = (w.v)(u.x) \tag{2.24}$$

or

$$w \otimes u..v \otimes x = (w.x)(u.v) \tag{2.25}$$

Note the order of these two operations, each of which is called a double-dot (or scalar) product [4].

Other important relations involving the transpose are

$$(\mathbf{S} + \mathbf{T})^T = \mathbf{S}^T + \mathbf{T}^T \tag{2.26}$$

$$\left(\mathbf{S}.\mathbf{T}\right)^{T} = \mathbf{T}^{T}\mathbf{S}^{T}, \tag{2.27}$$

$$\left(\mathbf{S}^{T}\right)^{T}, \tag{2.28}$$

and likewise for the trace,

$$tr\left(a.\mathbf{S} + b.\mathbf{T}\right) = atr(\mathbf{S}) + btr(\mathbf{T}), \tag{2.29}$$

$$tr\left(\mathbf{S}.\mathbf{T}\right) = tr(\mathbf{T}.\mathbf{S}), \tag{2.30}$$

$$tr(\mathbf{S}^{T}) = tr(\mathbf{S}), \tag{2.31}$$

And for the determinant,

$$\det(a\mathbf{S}) = a^{3}\det(\mathbf{S}), \tag{2.32}$$

$$\det(\mathbf{S}.\mathbf{T}) = \det(\mathbf{S})\det(\mathbf{T}), \tag{2.33}$$

$$\det(\mathbf{S}^{T}) = \det\left(\mathbf{S}\right). \tag{2.34}$$

Here, it should be noted that a tensor is said to be symmetric or skew-symmetric if, respectively,

$$\mathbf{U} = \mathbf{U}^{T}, \quad \mathbf{W} = -\mathbf{W}^{T}. \tag{2.35}$$

Every skew-symmetric tensor $\mathbf{W}$ has an associated axial vector w such that $\mathbf{W}.v = w \times v$ for all vectors v [4].

Further, every second order tensor $\mathbf{T}$ can be written as the sum of asymmetric tensor $\mathbf{U}$ and skew-symmetric tensor $\mathbf{W}$, that is,

$$\mathbf{T} = \mathbf{U} + \mathbf{W}, \text{ where } \mathbf{U} = \frac{1}{2}\left(\mathbf{T} + \mathbf{T}^{T}\right), \quad \mathbf{W} = \frac{1}{2}(\mathbf{T} - \mathbf{T}^{T}). \tag{2.36}$$

It is easy to show, therefore, that

$$tr(\mathbf{W}) = 0, \quad \det(\mathbf{W}) = 0. \qquad \textbf{(2.37)}$$

The square, cube, etc. of a tensor are given by

$$\mathbf{S}^2 = \mathbf{S.S}, \quad \mathbf{S}^3 = \mathbf{S.S}^2. \qquad \textbf{(2.38)}$$

There are two special second order tensors of importance, namely the zero tensor $\mathbf{O}$ and the identity tensor $\mathbf{I}$, where

$$\mathbf{O}.v = o, \quad \mathbf{I}.v = v. \qquad \textbf{(2.39)}$$

That is, the zero tensor transforms all vectors in to the zero vector and the identity tensor transforms all vectors into themselves. Likewise,

$$\mathbf{O.S} = \mathbf{O}, \quad \mathbf{I.S} = \mathbf{S}. \qquad \textbf{(2.40)}$$

The trace and the determinant of the identity tensor arise often. They are

$$tr(\mathbf{I}) = 3, \quad \det(\mathbf{I}) = 1 \qquad \textbf{(2.41)}$$

The inverse of a tensor $(\ldots)^{-1}$ is defined by

$$\mathbf{S.S}^{-1} = \mathbf{I}, \quad \mathbf{S}^{-1}.\mathbf{S} = \mathbf{I}. \qquad \textbf{(2.42)}$$

Important relations for the inverse are

$$(a.\mathbf{S})^{-1} = \frac{1}{a}\mathbf{S}^{-1}, \qquad \textbf{(2.43)}$$

$$(\mathbf{S.T})^{-1} = \mathbf{T}^{-1}.\mathbf{S}^{-1}. \qquad \textbf{(2.44)}$$

Moreover, the transpose and determinant of the inverse of a tensor are given by

$$\left(\mathbf{S}^{-1}\right)^T = \left(\mathbf{S}^T\right)^{-1}, \quad \det\left(\mathbf{S}^{-1}\right) = \frac{1}{\det \mathbf{S}}. \qquad \textbf{(2.45)}$$

Note, too, that $\left(\mathbf{S}^{-1}\right)^T$ is often denoted by $\mathbf{S}^{-T}$.

Finally, a second order tensor $\mathbf{Q}$ is called orthogonal if

$$\mathbf{Q}.\mathbf{Q}^T = \mathbf{Q}^T.\mathbf{Q} = I \qquad \textbf{(2.46)}$$

That is, if its inverse equals its transpose. Also, the equations above reveal that

$$\det(\mathbf{Q}) = \pm 1 . \qquad \textbf{(2.47)}$$

An orthogonal tensor is said to be proper if $\det(\mathbf{Q}) = 1$.

Because many operations on tensors take a special form depending on the type of tensor, it is often useful to introduce the following nomenclature: let Lin denote all second-order tensors,

All symmetric tensors, Psym all positive-definite symmetric tensors, Skw all skew-symmetric tensors, and Orth all orthogonal tensors with Orth^+ being those that are proper orthoganal. Hence, for example, $\mathbf{W} \in$ Skew implies that $\mathbf{W}$ is a skew-symmetric tensor. Vectors v and scalars a are similar If denoted by $v \in V$ and $a \in R$, which is to say that they are members of the vector space V or real numbers R, respectively [4].

2.1.2 Index Notation

So far algebra has been presented in symbolic notation exclusively employing bold face letters. It represents a very convenient and concise tool to manipulate most of the relations used in continuum mechanics. However, particularly in computational mechanics, it is essential to refer vector quantities to a basis. Additionally, to gain more insight in some quantities and to carry out mathematical operations among tensors more readily it is often helpful to refer to components [3].

In order to present coordinate expressions relative to a right-handed and orthonormal system we introduce a fixed set of three basis vectors e_1, e_2, e_3, (sometimes introduced as i, j, k) called a (Cartesian) basis, with properties

$$e_1.e_2 = e_1.e_3 = e_2.e_3 = 0, \quad e_1.e_1 = e_2.e_2 = e_3.e_3 = 1 \qquad \textbf{(2.48)}$$

These vectors of unit length which are mutually orthogonal form a so-called orthonormal system. Then any vector u in the three-dimensional Euclidean space is represented uniquely by a linear combination of the basis vectors e_1, e_2, e_3, i.e.

$$u = u_1 e_1 + u_2 e_2 + u_3 e_3 \ , \tag{2.49}$$

Where the three real numbers u_1, u_2, u_3 are the uniquely determined Cartesian components of vector u along the given directions e_1, e_2, e_3, respectively.

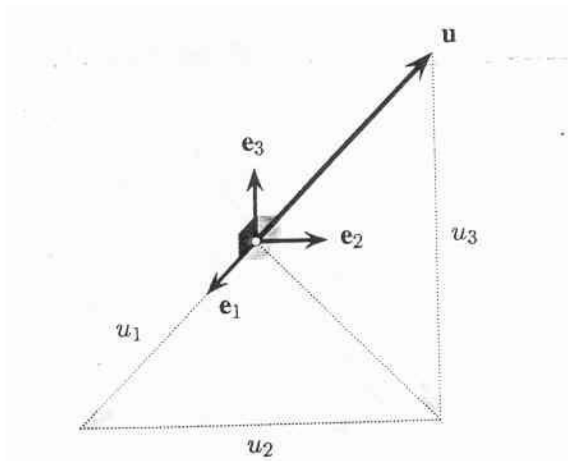


Figure 2.1: Vector u with its cartesian components u_1, u_2, u_3

Using index notation the vector *u* can be written as $u = \sum_{i=1}^{3} u_i e_i$ or in an abbreviated form by leaving out the summation symbol, simply as

$u = u_i e_i$, (sum over *i*=1,2,3) .

The summation convention says that whenever an index is repeated (only once) in the same term, then, a summation over the range of this index is implied unless otherwise indicated [3].

The index *i* that is summed over is said to be a dummy index, since a replacement by any other symbol does not affect the value of the sum. An index that is not summed over in a given term is called a free index. Note that in the same equation an index is either dummy or free. Thus, these relations can be written in a more convenient form as

$$e_i.e_j = \delta_{ij} \equiv \begin{cases} 1, \text{ if } \ \mathrm{i} = \mathrm{j} \\ 0, \text{if } \ \mathrm{i} \neq \mathrm{j} \end{cases} \tag{2.50}$$

Which defines the Kronecker delta δ_{ij}. The useful properties are

$$\delta_{ii} = 3, \ \ \delta_{ij}u_i = u_j, \ \ \delta_{ij}\delta_{jk} = \delta_{ik}. \tag{2.51}$$

Taking the basis $\{e_i\}$ and the equations above, the component expression for the dot product gives,

$$u.v = u_i e_i.v_j e_j = u_i v_j e_i e_j = u_i v_j \delta_{ij} = u_i v_i \tag{2.52}$$

$$u.v = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{2.53}$$

In an analogous manner, the component expression for the square of the length of *u*, i.e.

$$|u|^2 = u_1^{\ 2} + u_2^{\ 2} + u_3^{\ 2} \tag{2.54}$$

The cross product of *u* and *v*, denoted by $u \times v$ produces a new vector. In order to express the cross product in terms of components the permutation symbol is introduced as,

$$\varepsilon_{ijk} = \begin{cases} 1\ , & \text{for even permutations of (i, j, k)} \\ \text{-}1, & \text{for odd permutations of (i, j, k)}\ , \\ 0, & \text{if there is a repeated index} \end{cases} \tag{2.55}$$

Consider the right-handed and orthonormal basis $\{e_i\}$, then

$$e_1 \times e_2 = e_3, \quad e_2 \times e_3 = e_1, \quad e_3 \times e_1 = e_2, \tag{2.56}$$

$$e_2 \times e_1 = -e_3, \quad e_3 \times e_2 = -e_1, \quad e_1 \times e_3 = -e_2, \tag{2.57}$$

$$e_1 \times e_1 = e_2 \times e_2 = e_3 \times e_3 = 0 \tag{2.58}$$

Or in more convenient short-hand notation

$$e_i \times e_j = \varepsilon_{ijk} e_k . \tag{2.59}$$

Then the cross product of *u* and *v* yields,

$$w = u \times v = u_i e_i \times v_j e_j = u_i v_j \left(e_i \times e_j\right) = \varepsilon_{ijk} u_i v_j e_k = w_k e_k \tag{2.60}$$

Recall the components of the resultant vector *u* relative to the coordinate axes. That is,

$$u = u_1 + u_2 + u_3 = u_1 e_1 + u_2 e_2 + u_3 e_3 . \tag{2.61}$$

This equation also reveals that any vector can be represented in terms of linearly independent vectors. Likewise, any-second order tensor can be represented in terms of linearly independent dyads, as, for example, $e_1 \otimes e_1, e_1 \otimes e_2 \ldots$ in Cartesian components.

Where T_{11}, T_{12}, etc. are said to be components of **T** relative to Cartesian axes. The equation above can be written in the more compact Einstein summation convention as

$$\mathbf{T} = T_{ij} e_i \otimes e_j \tag{2.62}$$

where the subscripts i and j are both repeated, that is "dummy." Note, too, the nine components of **T** with respect to a Cartesian coordinate system, say T_{mn}, can easily be determined, viz.,

$$T_{mn} = e_m .(T_{ij} e_i \otimes e_j).e_n , \tag{2.63}$$

$$T_{mn} = T_{ij} (e_m .e_i)\left(e_j e_n\right), \tag{2.64}$$

$$T_{mn} = T_{ij} \delta_{mi} \delta_{jn} . \tag{2.65}$$

wherein we again factored out the scalar components T_{ij} before performing the dot products (on vectors); the replacement property of the Kronecker delta is thus revealed again. Because a second-order tensor has nine components, they can also be written in the form of a 3 x 3 matrix as

$$T_{ij} = [\mathbf{T}] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}. \qquad \textbf{(2.66)}$$

A familiar example of matrix representation is the identity tensor **I**, which has components

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \textbf{(2.67)}$$

Relative to cartesian coordinate axes. Thus, recalling the definition of Kronecker delta, we see the Kronecker delta simply represents the components of **I** relative to Cartesian coordinate system. That is, we can write

$$\mathbf{I} = \delta_{ij} e_i \otimes e_j \, . \qquad \textbf{(2.68)}$$

Cartesian component representations for vectors and tensors reveal that the transformatian of a vector into another vector via a second-order tensor simply involves a scalar product between appropriate bases:

$$\left(T_{ij} e_i \otimes e_j\right)\left(v_k e_k\right) = T_{ij} v_k e_i \left(e_j e_k\right), \qquad \textbf{(2.69)}$$

$$\left(T_{ij} e_i \otimes e_j\right)\left(v_k e_k\right) = T_{ij} v_k e_i \delta_{jk} \, , \qquad \textbf{(2.70)}$$

$$\left(T_{ij} e_i \otimes e_j\right)\left(v_k e_k\right) = T_{ij} v_k e_i \, , \qquad \textbf{(2.71)}$$

$$\left(T_{ij} e_i \otimes e_j\right)\left(v_k e_k\right) = u_i e_i \, , \qquad \textbf{(2.72)}$$

wherein we again used the replacement property of the Kronecker delta and let u_i represent the term(s) $T_{i1} v_1 + T_{i2} v_2 + T_{i3} v_3$. The equation above reveals that many tensor manipulations can be reduced to manipulations of the bases; thus, they are no more difficult than the vector operations learned in Engineering Static. Likewise, the transpose, trace, determinant, and inverse operations are straightforward, based on the rules given in equations before. For example,

$$\mathbf{T}^T = \left(T_{ij} e_i \otimes e_j\right)^T = T_{ij} e_j \otimes e_i , \tag{2.73}$$

$$tr\mathbf{T} = tr\left(T_{ij} e_i \otimes e_j\right) = T_{ij}\left(e_i . e_j\right) = T_{ii} , \tag{2.74}$$

$$\det \mathbf{T} = \det[\mathbf{T}] = \det\left(T_{ij}\right), \tag{2.75}$$

$$\mathbf{T}^{-1} = \left(T_{ij} e_i \otimes e_j\right)^{-1} = T_{ij}{}^{-1} e_j \otimes e_i \ . \tag{2.76}$$

In particular, note that in this direct notation, the transpose of a vector equals the vector itself, that is $v^T \equiv v$ or $\left(v_i e_i\right)^T \equiv v_i e_i$. Moreover, the inverse switches the order of the bases that constitute the dyad, just as the transpose does, but it also modifies the scalar components [4].

Next, consider a special vector called the del operator, which relative to Cartesian coordinates is defined by

$$\nabla = e_i \frac{\partial}{\partial x_i} \tag{2.77}$$

and from which we obtain, for example, the gradient of a scalar a,

$$\nabla a = e_i \frac{\partial}{\partial x_i}(a) = \frac{\partial a}{\partial x_i} e_i ; \tag{2.78}$$

The divergence and gradient of a vector *u*, that is,

$$\nabla u = e_i \frac{\partial}{\partial x_i}\left(u_j e_j\right) = e_i\left(\frac{\partial u_j}{\partial x_i} e_j + u_j \frac{\partial e_j}{\partial x_i}\right) = \frac{\partial u_j}{\partial x_i}\left(e_i e_j\right) = \frac{\partial u_i}{\partial x_i} , \tag{2.79}$$

and

$$\nabla u = e_i \frac{\partial}{\partial x_i}\left(u_j e_j\right) = \frac{\partial u_j}{\partial x_i} e_i \otimes e_j \,; \tag{2.80}$$

or the divergence tensor **T**,

$$\nabla.\mathbf{T} = e_k \frac{\partial}{\partial x_k}\left(T_{ij} e_i \otimes e_j\right), \tag{2.81}$$

$$\nabla.\mathbf{T} = \frac{\partial T_{ij}}{\partial x_k}\left(e_k e_i\right) e_j = \frac{\partial T_{ij}}{\partial x_i} e_j \ . \tag{2.82}$$

Hence, ∇u yields a scalar, ∇a and $\nabla.\mathbf{T}$ yield vectors, and ∇u yields a tensor.

Here, it is important to reemphasize that tensor components are scalar quantities; hence, the associative laws are very useful in the preceding derivations. Moreover, the orthonormal Cartesian bases e_i are independent of position **x**, which is why their derivatives (due to the product rule for differentiation) disappeared in these operations; we will see that the situation is different for curvilinear coordinate systems wherein the bases can depend on position [4].

Another convention arises naturally when one takes a derivative with respect to a vector.

$$\frac{\partial a}{\partial \mathbf{x}} \equiv \frac{\partial a}{\partial \mathbf{x}_i} e_i \tag{2.83}$$

And

$$\frac{\partial u}{\partial \mathbf{x}} = \frac{\partial\left(u_i e_i\right)}{\partial \mathbf{x}} = \frac{\partial\left(u_i e_i\right)}{\partial \mathbf{x}_j} \otimes e_j = \frac{\partial u_i}{\partial \mathbf{x}_j} e_i \otimes e_j \ . \tag{2.84}$$

Derivatives with respect to a second-order tensor follow a similar convention:

$$\frac{\partial a}{\partial \mathbf{T}} = \frac{\partial a}{\partial T_{ij}} e_i \otimes e_j \ . \tag{2.85}$$

The scalar products between two second-order tensors are

$$\mathbf{T}:\mathbf{S} = \left(T_{ij} e_i \otimes e_j\right):\left(S_{mn} e_m \otimes e_n\right) \ , \tag{2.86}$$

$$\mathbf{T}:\mathbf{S} = T_{ij} S_{mn} \left(e_i e_m\right)\left(e_j e_n\right) \ , \tag{2.87}$$

$$\mathbf{T}:\mathbf{S} = T_{ij} S_{mn} \delta_{im} \delta_{jn} \ , \tag{2.88}$$

$$\mathbf{T}:\mathbf{S} = T_{ij} S_{ij} \ , \tag{2.89}$$

2.1.3 Coordinate Transformations

It is worthwhile to mention that vectors and tensors themselves remain invariant upon a change of basis – they are said to be independent of any coordinate system. However, their respective components do depend upon the coordinate system introduced, which is arbitrary. The components change their magnitudes by a rotation of the basis vectors, but are independent of any translation.

We now set up the transformation laws for various components of vectors and tensors under a change of basis.

$$\tilde{e}_i = Q e_i \quad \text{and} \quad e_i = Q^T \tilde{e}_i \ , \quad i = 1,2,3 \tag{2.90}$$

where Q denotes the orthogonal tensor, with components Q_{ij} which are the same in either basis. The components describe the orientation of the two sets of basis vectors relative to each other. In particular, $\mathbf{Q}$ rotates the basis vectors e_i in to $\tilde{e}_i$, while $\mathbf{Q}^T$ rotates $\tilde{e}_i$ back to e_i. Using equations (2 63) and ((2 50)) we find that

$$\mathbf{Q} e_i = Q_{ij} e_j \quad \textbf{and} \quad \mathbf{Q}^T \tilde{e}_i = Q_{ij} \tilde{e}_j \ , \tag{2.91}$$

By comparing the equations above we may extract the orthogonality condition of the cosines, characterized by $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{I}$. Equivalently, expressed in index or matrix notation

$$Q_{ij} Q_{ik} = Q_{ji} Q_{ki} = \delta_{jk} \ ,$$

$$Q_{ij}Q_{ik} = Q_{ji}Q_{ki} = \delta_{jk} \ , \qquad \textbf{(2.92)}$$

Where [**Q**] contains the collection of the components Q_{ij}. It is an orthogonal matrix which is referred to as the transformation matrix. Note that $[\mathbf{Q}]^{T}$ = $[\mathbf{Q}^{T}]$. In order to maintain the right-handedness of the basis vectors we have admitted only rotations of the basis vectors, consequently $\det[\mathbf{Q}] = \pm 1$ [3].

2.1.4 Vectorial Transformation Law

We consider any vector u resolved along the two sets $\{\tilde{e}_i\}$ and $\{e_i\}$ of basis vectors, i.e.

$$\tilde{u}_i = \mathbf{u}.\tilde{e}_i \quad \text{in} \quad \{\tilde{e}_i\} \ , \qquad \textbf{(2.93)}$$

$$u_i = \mathbf{u}.e_i \quad \text{in} \quad \{e_i\} \ . \qquad \textbf{(2.94)}$$

We obtain the vectorial transformation law for the Cartesian components of the vector **u**, i.e.

$$\tilde{u}_i = \mathbf{u}.\tilde{e}_i = Q_{ji}\left(u.e_j\right) = Q_{ji}u_j \qquad \textbf{(2.95)}$$

$$\text{and} \quad [\tilde{\mathbf{u}}] = [\mathbf{Q}]^{T}[\mathbf{u}] . \qquad \textbf{(2.96)}$$

These equations determine the relationship between the components of a vector associated with the (old) basis $\{e_i\}$ and the components of the same vector associated with another (new) basis $\{\tilde{e}_i\}$ [3].

2.1.5 Tensorial Transformation Law

To determine the transformation laws for the Cartesian components of any second-order tensor **A**, we describe its components along the sets $\{\tilde{e}_i\}$ and $\{e_i\}$ of basis vectors, i.e.

$$\tilde{A}_{ij} = \tilde{e}_i.A\tilde{e}_j \quad \text{in} \quad \{\tilde{e}_i\} \ , \qquad \textbf{(2.97)}$$

$$A_{ij} = e_i.Ae_j \quad \text{in} \quad \{e_i\} \ . \qquad \textbf{(2.98)}$$

Combining the equations above with (*(2 63)) and (2 92), then the components A_{ij}, $\tilde{A}_{ij}$ are related via the so-called tensorial transformation law.

$$\tilde{A}_{ij} = \tilde{e}_i.\mathbf{A}\tilde{e}_j = (Q_{ki}e_k).\mathbf{A}(Q_{mj}e_m), \qquad \textbf{(2.99)}$$

$$\tilde{A}_{ij} = Q_{ki}Q_{mj}(e_k.\mathbf{A}e_m), \qquad \textbf{(2.100)}$$

$$\tilde{A}_{ij} = Q_{ki}Q_{mj}A_{km} \qquad \textbf{(2.101)}$$

or

$$[\tilde{\mathbf{A}}] = [\mathbf{Q}]^T[\mathbf{A}][\mathbf{Q}] \ . \qquad \textbf{(2.102)}$$

Transformation $[\tilde{\mathbf{A}}] = [\mathbf{Q}]^T[\mathbf{A}][\mathbf{Q}]$ relates different matrices $[\tilde{\mathbf{A}}]$ and $[\mathbf{A}]$, which have the components of the same tensor **A**. In analogous manner, we find that

$$A_{ij} = Q_{ki}Q_{jm}\tilde{A}_{km} \quad \text{or} \quad [\mathbf{A}] = [\mathbf{Q}]^T[\tilde{\mathbf{A}}][\mathbf{Q}]. \qquad \textbf{(2.103)}$$

2.1.6 Principal Values

The scalars λ_i characterize eigenvalues of a tensor **A** if there exist corresponding nonzero normalized eigenvectors $\hat{n}_i$ of **A**, so that

$$\mathbf{A}\hat{n}_i = \lambda_i\hat{n}_i \ , \quad (i = 1, 2, 3; \text{no summation}) \qquad \textbf{(2.104)}$$

To identify the eigenvectors of a tensor, we use subsequently a hat on the vector quantity concerned, for example $\hat{\mathbf{n}}$.

Thus, a set of homogeneous algebraic equations for the unknown eigenvalues λ_i, i = 1, 2, 3, and the unknown eigenvectors $\hat{n}_i$, i = 1, 2, 3 is

$$(\mathbf{A} - \lambda_i\mathbf{I})\hat{n}_i = 0 \ , \ (i = 1, 2, 3; \text{no summation}) \ . \qquad \textbf{(2.105)}$$

Eigenvalues characterize the physical nature of a tensor. They do not depend on coordinates. For a positive definite symmetric tensor **A**, all eigenvalues λ_i are (real and) positive since, using **(**2-106**),** we have $\lambda_i = \hat{n}_i . \mathbf{A}\hat{n}_i > 0$, i = 1,2,3. Moreover, the set of eigenvectors of a symmetric tensor **A** form a mutually orthogonal basis $\{\hat{n}_i\}$ [3].

2.1.7 Principal Scalar Invariants

For the system (2 106) to have solutions $\hat{n}_i \neq 0$ the determinant of the system must vanish. Thus,

$$\det(\mathbf{A} - \lambda_i \mathbf{I}) = 0 \ , \tag{2.107}$$

where.

$$\det(\mathbf{A} - \lambda_i \mathbf{I}) = -\lambda_i^{\ 3} + I_1 \lambda_i^{\ 2} - I_2 \lambda_i + I_3 \ . \tag{2.108}$$

This requires that we solve a cubic equation in λ, usually written as

$$\lambda_i^{\ 3} - I_1 \lambda_i^{\ 2} + I_2 \lambda_i - I_3 = 0 \tag{2.109}$$

called the characteristic polynomial (or equation) for **A**, the solutions of which are

the eigenvalues λ_i, $i = 1,2,3$. **(2.110)**

Here, $I_i(\mathbf{A})$, i = 1,2,3, are the so-called principal scalar invariants of **A**. In terms of **A** and its principal values λ_i, *i* = 1,2,3, these are given by

$$I_1(\mathbf{A}) = A_{ii} = tr\mathbf{A} = \lambda_1 + \lambda_2 + \lambda_3 \ , \tag{2.111}$$

$$I_2(\mathbf{A}) = \frac{1}{2}\left(A_{ii}A_{jj} - A_{ji}A_{ij}\right) = \frac{1}{2}[(tr\mathbf{A})^2 - tr(\mathbf{A}^2)] = tr\mathbf{A}^{-1} \det \mathbf{A} \ , \tag{2.112}$$

$$I_3(\mathbf{A}) = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k} = \det \mathbf{A} = \lambda_1 \lambda_2 \lambda_3 \ . \tag{2.113}$$

A repeated application of tensor **A** to eq. (2 105) yields $\mathbf{A}^{\alpha}\hat{n}_i = \lambda_i^{\alpha}\hat{n}_i$, i = 1,2,3, for any positive integer α. Using this relation and (2 109) multiplied by $\hat{n}_i$, we obtain the well-known Cayley-Hamilton equation

$$\mathbf{A}^3 - I_1\mathbf{A}^2 + I_2\mathbf{A} - I_3\mathbf{I} = \mathbf{0} \qquad \textbf{(2.114)}$$

It states that every (second-order) tensor **A** satisfies its own characteristic equation [3].

2.1.8 Spectral Decomposition of a Tensor

Any symmetric tensor **A** may be represented by its eigenvalues λ_i, i = 1,2,3, and the corresponding eigenvectors of **A** forming an orthonormal basis $\{\hat{n}_i\}$. Using the unit tensor, by analogy with (2 105), i.e. $\mathbf{I} = \hat{n}_i \otimes \hat{n}_i$ and elations (2 50), (2-115) we obtain an expression which is known as the spectral decomposition of **A**, i.e.

$$\mathbf{A} = \mathbf{AI} = \left(\mathbf{A}\hat{n}_i\right) \otimes \hat{n}_i = \sum_{i=1}^{3} \lambda_i \hat{n}_i \otimes \hat{n}_i \qquad \textbf{(2.116)}$$

The components A_{ij} of tensor **A** relative to a basis of principal directions follow with (2 63) by replacing e_i with the three orthonormal basis vectors $\{\hat{n}_i\}$. With equations (**2**.**117**), (2 50) we obtain

$$A_{ij} = \hat{n}_i.\mathbf{A}\hat{n}_j = \hat{n}_i.\lambda_j\hat{n}_j = \lambda_j\delta_{ij}\ , \quad (j = 1, 2, 3;\ \text{no summation}), \qquad \textbf{(2.118)}$$

which produces a diagonal matrix [**A**] in the form,

$$[\mathbf{A}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \qquad \textbf{(2.119)}$$

Where the diagonal elements are the eigenvalues of **A**. This result may be obtained directly from the spectral decomposition (2 115) of A [3].

2.1.9 Further Results in Tensor Calculus

Because vectors and tensor are defined on linear vector spaces, rules for differentiation are similar to those from elementary calculus. For example, if scalar, vector and tensor fields – say, $a \in R$ and $u, v \in V$, and $\mathbf{S}, \mathbf{T} \in Lin$ - depend only on the variable $t \in R$, then

$$\frac{d}{dt}(av) = \frac{da}{dt}v + a\frac{dv}{dt}, \tag{2.120}$$

$$\frac{d}{dt}(uv) = \frac{du}{dt}v + u\frac{dv}{dt}, \tag{2.121}$$

$$\frac{d}{dt}(\mathbf{T}v) = \frac{d\mathbf{T}}{dt}v + \mathbf{T}\frac{dv}{dt} \ , \tag{2.122}$$

And

$$\frac{d}{dt}(\mathbf{TS}) = \frac{d\mathbf{T}}{dt}\mathbf{S} + \mathbf{T}\frac{d\mathbf{S}}{dt} \ . \tag{2.123}$$

Similarly, it is useful to record the following identities:

$$\nabla(u \otimes v) = (\nabla u).v + u.\nabla v \tag{2.124}$$

And

$$\nabla(\mathbf{S}u) = (\nabla\mathbf{S})u + \mathbf{S} : (\nabla u) \tag{2.125}$$

Finally, we record the divergence theorem, which will be used extensively in the formulation of the five basic postulates of continuum mechanics. It is

$$\int\int (n\mathbf{T})da = \int\int\int (\nabla\mathbf{T})dv \ , \tag{2.126}$$

Where *da* and *dv* are differential areas and volumes, respectively, and n is an outward unit normal vector to *da*. In Cartesian components, the divergence theorem is

$$\int\int(n_i\mathbf{T}_{ij})da = \int\int\int\left(\frac{\partial\mathbf{T}_{ij}}{\partial x_i}\right)dv\,. \qquad \textbf{(2.127)}$$

In summary, albeit sometimes intimidating at first, tensor analysis is often no more difficult than vector analysis since all operations involve the base vectors. It is for this reason, therefore, that the dyadic approach is superior to the "classical component approach" to tensor analysis wherein one simply employs a complex set of rules and conventions to manipulate the components. In addition, although Cartesian component representations are only useful in certain boundary value problems, it is often easiest to derive tensoral relations using Cartesian components. Once finished, the results can be put into direct notation, in which they hold in general [3].

2.2 Kinematics

Kinematics is defined as the study of motion. However, motion not only includes the current movement of a body, but also how the position of a particle within a particular configuration of a body has changed relative to its position in reference configuration. . Here, we define a body to be a collection of material particles and configuration of the body to be the specification of the positions of each of the particles in the body at a particular time *t*. Motion can be defined, therefore, as a sequence of configurations parameterized by time [4].

It will prove useful to locate a generic particle in a reference configuration β_0, at time *t* = 0, via a position vector **X**, and likewise the position of the same particle in a current configuration β_t, at time *t*, via a position vector **x**. Although the reference configuration is often taken to be a stress-free, undeformed configuration, it need not be. It is also useful to refer **X** and **x** to different coordinate systems (that are related by a known translation and rotation): for Cartesian components, we refer **X** and **x** to the coordinate systems $\{\mathrm{O};\mathbf{E}_A\}$ and $\{\mathrm{o};\mathbf{e}_i)$, respectively. Hence, the position vectors have representations $\mathbf{X} = X_A\mathbf{E}_A$ and $\mathbf{x} = x_i\mathbf{e}_i$, where summation is implied over dummy indices A = 1,2,3 and i = 1,2,3 in E^3. Without a loss of generality, however, we will let the origins O and o coincide (Figure 2.2). The displacement vector $\boldsymbol{u}$ for each material particle is thus given by $\mathbf{u} = \mathbf{x} - \mathbf{X}$. With the exception of a rigid body motion, each particle constituting a body can experience a different displacement [4].

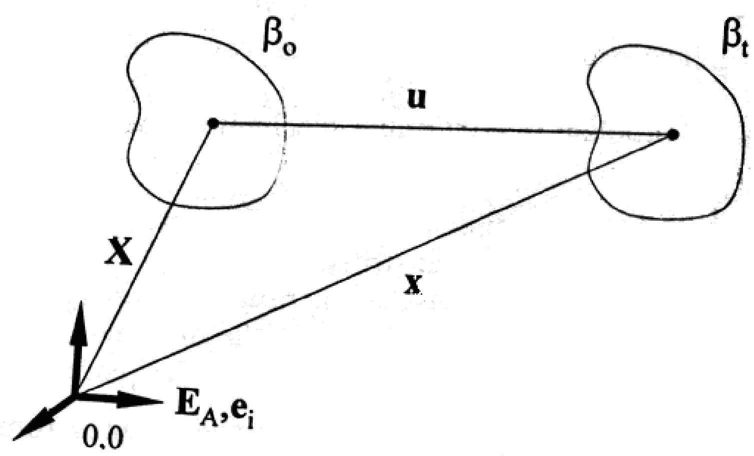


Figure 2.2: Schematic illustration of material body in two configurations

an initial reference configuration at time t = 0, denoted as β_0, and a current configuration at time t, denoted as β_t The position of a material particle, relative to a common arigin, is given by X and x in these two configurations, respectively. The displacement $\mathbf{u} = \mathbf{x} - \mathbf{X}$ and $\mathbf{E}_A$ and $\mathbf{e}_i$ are orthonormal bases [4].

There are four basic approaches to describe the kinematics of a continuum: the material, referential, spatial and relative approaches. In the material approach, motion is described via the particles themselves and time; this approach is not particularly useful in solid mechanics [4].

The Lagrangian (referential) description is a characterization of the motion with respect to the material coordinates (X_1, X_2, X_3) and time t. In material description attention is paid to a particle, and we observe what happens to the particle as it moves. Traditionally, the material description is often referred to as the Lagrangian description. Note that at t=0 we have the consistency condition $\mathbf{X} = \mathbf{x}$ and $X_A = x_a$.

The Eulerian (spatial) description is a characterization of the motion with respect to the spatial coordinates (x_1, x_2, x_3) and time t. In spatial description attention is paid to a point in space, and we study what happens at the point as the time changes.

In fluid mechanics we quite often work in the Eulerian description in which we refer all relevant quantities to the position in space at time *t*. It is not useful to refer the quantities to the material coordinates X_A, A = 1, 2, 3, at t=0, which are, in general, not known in fluid mechanics. However, in solid mechanics we use both types of description. Due to the fact that the constitutive behavior of solids is often given in terms of material coordinates we often prefer the Lagrangian description [3].

Finally, in the relative approach one uses independent variables $(\mathbf{x}, \tau)$ where τ is a measure of time often related to an intermediate configuration; this approach is useful in viscoelasticity [4].

Let the positions of material particles at time t depend on their original positions, v is,

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t), \ \mathbf{x}, \mathbf{X} \in V, \ t \in R \tag{2.128}$$

Hence the associated displacement field is given by,

$$\mathbf{u}(\mathbf{X}, t) = x(\mathbf{X}, t) - \mathbf{X} \tag{2.129}$$

Because we will be interested primarily in the motion of individual material particles, it is useful to consider what happens to generic differential line segments as a body passes from one configuration to another. Hence, let d***x*** be an oriented differential line segment in β_t that was originally d***X*** in β_0. A fundamental question then is how do we relate these two differential position vectors? Recall that a second order tensor transforms a vector in to a new vector. Hence in direct and Cartesian component notations, at each time *t*, let

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}, \qquad dx_i = F_{iA} \cdot dX_A \ . \tag{2.130}$$

Where F is a second order tensor that accomplishes the desired transformation. The quantify **F** is crucial in nonlinear continuum mechanics and is primary measure of deformation, called the deformation gradient. In general **F** has nine components for all *t*, and characterizes the behavior of motion in the neighborhood of a point.

Expression (2 128) clearly defines a linear transformation which generates a vector d***x*** by the action of the second order tensor **F** on the vector d***X***. Hence, equation (2 128) serves as transformation rule. Therefore, **F** is said to be a two point tensor involving points in two distinct configurations. One index describes spatial coordinates, x_a and the other material coordinates, X_A. In summary: material tangent vectors map (i.e. transform) in to spatial tangent vectors via the deformation gradient. [3].

Because ***x*** is a function of ***X***, at each fixed time t, the chain rule requires

$$\mathbf{dx} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \cdot \mathbf{dX} \ , \quad dx_i = \frac{\partial x_i}{\partial X_A} \cdot dX_A \tag{2.131}$$

Moreover, comparing equations above reveals that

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = F_{iA} \mathbf{e}_i \otimes \mathbf{E}_A , \tag{2.132}$$

where

$$F_{iA} = \frac{\partial x_i}{\partial X_A} \ , \tag{2.133}$$

This provides a method for computing the components F given a referential description of the motion relative to a Cartesian coordinate system [3].

Assuming equation (2 128) is invertible, that is X can be written as a function of x at a fixed time t, we can alternatively consider

$$\mathbf{dX} = \frac{\partial \mathbf{X}}{\partial \mathbf{x}} \cdot \mathbf{dx} , \quad dX_A = \frac{\partial X_A}{\partial x_i} dx_i \tag{2.134}$$

With

$$\mathbf{F}^{-1} = \frac{\partial \mathbf{X}}{\partial \mathbf{x}} = F^{-1}{}_{Ai} \mathbf{E}_A \otimes \mathbf{e}_i \ , \tag{2.135}$$

$$F_{Ai}^{-1} = \frac{\partial X_A}{\partial x_i}$$ **(2.136)**

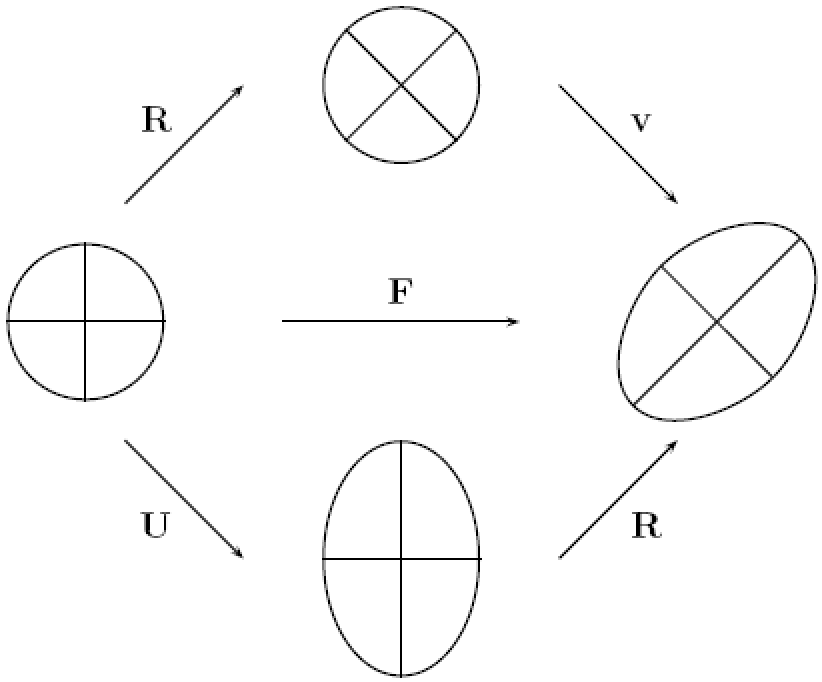


Figure 2.3: Decomposition in rotational and stretching part

It is important to observe that position vectors d$\boldsymbol{x}$ can be mapped from d$\boldsymbol{X}$ via a rigid body motion (i.e., a translation and/or rotation), a "deformation" (i. e., extension and shear), or a combination of both. Indeed, it can be shown that $\mathbf{F}$ can be decomposed via

$$\mathbf{F} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R}$$ **(2.137)**

where $\mathbf{R} \in Orth^{+}$ (i.e., $\mathbf{R}^{-1} = \mathbf{R}^{T}$ and $\det \mathbf{R} = 1$) represents the rigid body motion, $\mathbf{U} \in Psym$ (i.e., $\mathbf{U}^{T} = \mathbf{U}$ and is positive definite) is defined in the reference configuration β_0, and $\mathbf{V} \in Psym$ is defined in the current configuration β_t. Referred to Cartesian coordinates,

$$\mathbf{R} = R_{iA} \mathbf{e}_i \otimes \mathbf{E}_A = R_{Ai} \mathbf{E}_A \otimes \mathbf{e}_i \ ,$$ **(2.138)**

$$\mathbf{U} = U_{AB} \mathbf{E}_A \otimes \mathbf{E}_B , \quad \mathbf{V} = V_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \ .$$ **(2.139)**

Hence, **R** is a two-point tensor, whereas **U** and **V** are one-point tensors. **U** and **V** represent the complete deformation (extension and shear), but are called right and left "stretch" tensors, respectively, because their principal values are the principal stretches (e.g., current divided by reference lengths) experienced by the body at a point. Equation (2 135) can be interpreted, therefore, as "stretch" followed by a "rigid rotation" ($\mathbf{R}\cdot\mathbf{U}$) or a "rigid rotation" followed by "stretch" ($\mathbf{V}\cdot\mathbf{R}$); it is called the polar decomposition theorem.

At first, we will introduce the right Cauchy-Green tensor C defined by

$$\mathbf{C} = \mathbf{F}^T\mathbf{F}\,. \qquad \textbf{(2.140)}$$

C is symmetric and positive definite and, therefore, holds

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \left(\mathbf{F}^T\mathbf{F}\right)^T = \mathbf{C}^T\,. \qquad \textbf{(2.141)}$$

Further, we will define

$$\det\mathbf{C} = \left(\det\mathbf{F}\right)^2 = j^2 > 0 \qquad \textbf{(2.142)}$$

with J as the determinant of F called the volume ratio.

A commonly used strain measurement is the Green-Lagrange strain tensor E defined by

$$\mathbf{E} = \left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right) = \frac{1}{2}\left(\mathbf{C} - \mathbf{I}\right) \qquad \textbf{(2.143)}$$

which is based on the observation of the change of squared lengths of line elements. Since **C** and **1** are symmetric **E** is also symmetric. **C** and **E** are defined on the undeformed reference configuration and are, therefore, referred to as material strain tensors.

An important strain measure in terms of spatial coordinates is the left Cauchy-Green tensor b defined by

$$\mathbf{b} = \mathbf{F}\mathbf{F}^T \qquad \textbf{(2.144)}$$

The second order tensor **b** is as **C** symmetric and positive definite

$$\mathbf{b} = \mathbf{F}\mathbf{F}^T = \left(\mathbf{F}\mathbf{F}^T\right)^T = \mathbf{b}^T \qquad \textbf{(2.145)}$$

It can also be shown that

$$\det \mathbf{b} = \left(\det \mathbf{F}\right)^2 = j^2 > 0 \qquad \textbf{(2.146)}$$

holds.

An observation of the change of squared lengths of line elements defined in the current configuration leads to the spatial counterpart of E, namely the Euler-Almansi strain tensor e defined by

$$\mathbf{e} = \frac{1}{2}\left(\mathbf{I} - \mathbf{F}^{-T}\mathbf{F}^{-1}\right) = \frac{1}{2}(\mathbf{I} - \mathbf{b}^{-1}) \qquad \textbf{(2.147)}$$

It can be shown that the right Cauchy-Green tensor **C** and left Cauchy-Green tensor **b** can be expressed as

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^T\mathbf{R}^T\mathbf{R}\mathbf{U} = \mathbf{U}^2 \qquad \textbf{(2.148)}$$

And

$$\mathbf{b} = \mathbf{F}\mathbf{F}^T = \mathbf{V}\mathbf{R}\mathbf{R}^T\mathbf{V}^T = \mathbf{V}^2 \qquad \textbf{(2.149)}$$

C and **b** are both one-point, symmetric tensors that are independent of rigid body motion, **C** being defined in the reference configuration β_0 and **b** in the current configuration β_t. When referred to Cartesian coordinates,

$$\mathbf{C} = C_{AB}\mathbf{E}_A \otimes \mathbf{E}_B \ , \qquad \textbf{(2.150)}$$

where

$$C_{AB} = \frac{\partial x_i}{\partial X_A}\frac{\partial x_i}{\partial X_B} \ , \qquad \textbf{(2.151)}$$

$$\mathbf{b} = b_{ij}\mathbf{e}_i \otimes \mathbf{e}_j \ , \tag{2.152}$$

where

$$b_{ij} = \frac{\partial x_i}{\partial X_A}\frac{\partial x_j}{\partial X_A} \tag{2.153}$$

Next, it will prove useful to consider velocity, acceleration, velocity gradients, strain-rates, etc., that is, measures of time-dependent motions experienced by material particles within the body of interest. Indeed, one of the fundamental laws of mechanics, Newton's second law, relates accelerations to the forces that cause them in special (i.e., inertial) frames of reference.

Quantifying accelerations and associated measures is thereby fundamental to mechanics.

Simply put, velocity **v** is the time rate-of-change of position, and acceleration **a** is the time rate-of-change of velocity; both are vectors. Recall, too, that we can use either a referential or a spatial description of motion. The referential description is the most intuitive, and typically the one used in dynamics and solid mechanics. In this approach, we let the current position **x** of a material particle depend on the reference position **X** and time *t*. Consequently,

$$\mathbf{v}(t) = \frac{d}{dt}(\mathbf{x}(\mathbf{X},t)) = \frac{d}{dt}(\mathbf{u}(\mathbf{X},t)) \tag{2.154}$$

And

$$\mathbf{a}(t) = \frac{d^2}{dt^2}(\mathbf{x}(\mathbf{X},t)) = \frac{d^2}{dt^2}(\mathbf{u}(\mathbf{X},t)) \tag{2.155}$$

where $\mathbf{u} = \mathbf{x}(\mathbf{X},t) - \mathbf{X}$; being the reference position, **X** does not change in time. Thus, referential **v** and **a**, for a given particle, depend only on time and their original position. In contrast, in a spatial approach, which has independent variables **x** and *t* and is typically used in fluid mechanics, we have

$$v(\mathbf{x},t) = \frac{d\mathbf{x}}{dt} \quad , \quad \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial t} \tag{2.156}$$

Where in we see that in the, spatial approach the acceleration has two contributions, one local $(\partial \mathbf{v}/\partial t)$ and one convective. The latter can be written as $\mathbf{L}\cdot\mathbf{v}$, where $\mathbf{L}(= \partial\mathbf{v}/\partial\mathbf{x})$ is called the velocity gradient tensor. Recalling that a second-order tensor transforms one vector into another, **L** transforms the differential position vector d**x** into the velocity vector d**v** at each time *t* (i.e., $d\mathbf{v} = \mathbf{L}\cdot d\mathbf{x}$, which shows that a tensor can change not only the magnitude and direction of a vector, but also its physical unit). **L** is thus a one point tensor defined in β_t. Although **L** arises naturally, it is not symmetric in general. Recalling equation (2 36), however, we can write

$$\mathbf{L} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T\right) + \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^T\right) = \mathbf{D} + \mathbf{W} \tag{2.157}$$

Where $\mathbf{D} \in Sym$, and is called the stretching tensor, and $\mathbf{W} \in Skw$ ($\mathbf{W} = -\mathbf{W}^T$), and is called the spinning (or sometimes vorticity) tensor. **L** and **D** play important roles in the first and second laws of continuum thermomechanics. In cartesian components,

$$\mathbf{D} = D_{ij}\mathbf{e}_i \otimes \mathbf{e}_j \text{, where} \quad D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{2.158}$$

Although, **D** is sometimes called the strain-rate tensor, it can be shown that

$$d\mathbf{E}/dt = \mathbf{F}^{\mathbf{T}} \cdot \mathbf{D} \cdot \mathbf{F} \tag{2.159}$$

Where **E** is the Green strain tensor. Hence, only if $\mathbf{F} \cong \mathbf{I}$, which is to say $\mathbf{E} \cong \boldsymbol{\varepsilon}$, may have

$$d\boldsymbol{\varepsilon}/dt \cong \mathbf{D}. \tag{2.160}$$

Because **F** fundamental measure of deformation, it is always useful to ask how it is related to all other kinematical quantities. Note, therefore that

$$\frac{d\mathbf{F}}{dt} = \frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right) = \frac{\partial}{\partial \mathbf{X}}\left(\frac{d\mathbf{x}}{dt}\right) = \frac{\partial \mathbf{v}}{\partial \mathbf{X}} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \tag{2.161}$$

Which is to say that $d\mathbf{F}/dt = \mathbf{L} \cdot \mathbf{F.A}$. A similar result exists for the time rate of change $\mathbf{F}^{-1}$. That is, the velocity gradient can be found from **F**, viz.,

$$\mathbf{L} = \frac{d\mathbf{F}}{dt} \cdot \mathbf{F}^{-1}, \qquad \mathbf{L} = -\mathbf{F} \cdot \frac{d\mathbf{F}^{-1}}{dt} \tag{2.162}$$

Later we will find that dJ/dt, where $J = \det \mathbf{F}$, plays a key role in the fundamental balance equations (e.g., conservation of mass, momentum and energy). Hence, note that

$$\frac{d}{dt}(\det \mathbf{F}) = \frac{\partial}{\partial \mathbf{F}}(\det \mathbf{F}) : \frac{d\mathbf{F}}{dt} \tag{2.163}$$

Which by using equations 2.93 and 3.30 , becomes

$$\frac{dJ}{dt} = \left(\det \mathbf{F}\mathbf{F}^{-T}\right) : (\mathbf{L} \cdot \mathbf{F}) \tag{2.164}$$

Which, in turn, can be written as (using eq 2.68)

$$\frac{dJ}{dt} = (\det \mathbf{F}) tr(\mathbf{F}^{-T} \cdot \mathbf{F}^{T} \cdot \mathbf{L}^{T}) = J tr \mathbf{L} \tag{2.165}$$

Where it is easy to show that $tr\mathbf{L} = \nabla \cdot \mathbf{v}$, that is the divergence of the velocity field [16].

2.3 The Concept Of Stress

Force is considered by many to be an intuitive concept, yet its precise definition is not necessarily straightforward. Nonetheless, we consider force to simply be the action of one body on another, which is a vectorial push or pull. There are two general types of forces that we shall be interested in: body forces, such as gravity or electromagnetic forces, which act on all material particles in a body without physical contact, and surface forces, such as a pressure or frictional forces, which act through physical contact on a body through its bounding surface.

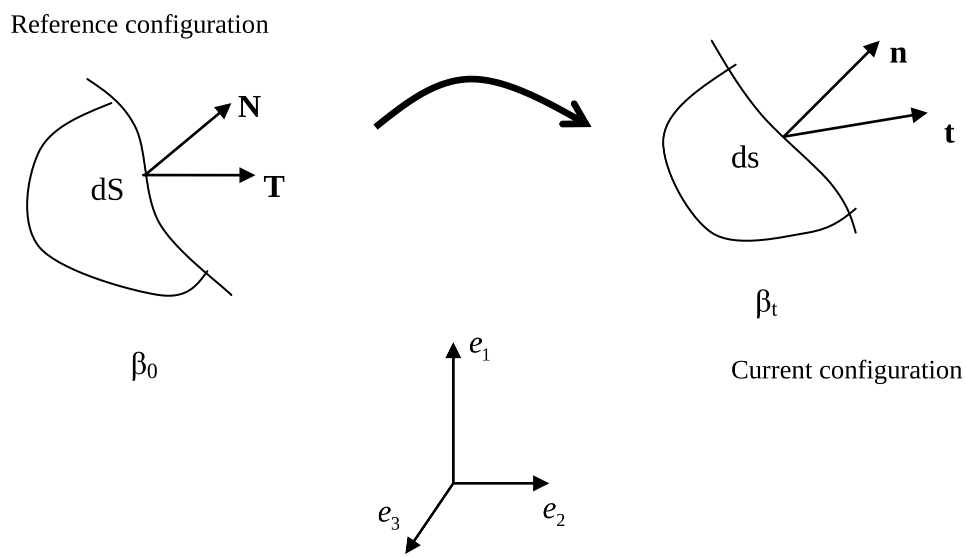


Figure 2.4: Traction vectors

Because many forces act on a material through a surface area, it is very useful to define a traction vector $\mathbf{T}^{(n)}$ as follows;

$$\mathbf{T}^{(n)} = \lim_{\Delta a \to 0}\left(\frac{\Delta f}{\Delta a}\right) = \frac{df}{da} \tag{2.166}$$

Where df is a differential force vector and da a differential area, both defined in β_t, with da having an orientation given by the outward unit normal vector n. That $df = \mathbf{T}^{(n)} da$ will prove convenient below in the definition of multiple measure of stress.

Now we introduce Cauchy's postulate of the form

$$\mathbf{t} = \mathbf{t}(x, t, n) \quad \mathbf{T} = \mathbf{T}(X, t, N) \tag{2.167}$$

there **t** represents the Cauchy traction vector (force measured per unit surface area defined in the current configuration) with n as the unit outward normal. The vector **T** represents the first Piola-Kirchhoff traction vector which represents a force measured per unit surface area defined in the reference configuration. The pseudo traction vector **T** does not describe the actual intensity. It acts on the region **t**, i.e. the current configuration, but it is described in terms of the reference position X and the outward normal **N** to the boundary surface β_0.

Using the introduced traction vectors we can write Cauchy's stress theorem as

$$\mathbf{t}(x,t,n)=\boldsymbol{\sigma}(x,t)\mathbf{n} \tag{2.168}$$

$$\mathbf{T}(X,t,N)=\mathbf{P}(X,t)\mathbf{N} \tag{2.169}$$

There _ denotes a symmetric (see balance laws) spatial tensor field called the Cauchy stress tensor. **P** is called the first Piola-Kirchhoff stress tensor. Further, we can introduce Newton's third law of action and reaction of the form

$$\mathbf{t}(x,t,n)=-\mathbf{t}(x,t,-n) \tag{2.170}$$

$$\mathbf{T}(X,t,N)=-\mathbf{T}(X,t,-N) \tag{2.171}$$

Now we can derive a relation between $\boldsymbol{\sigma}$ and **P** using Nanson's formula connecting line elements in the different configurations

$$da=J\mathbf{F}^{-T}dA \tag{2.172}$$

which leads to

$$\mathbf{t}(x,t,n)da=\mathbf{T}(X,t,N)dA, \tag{2.173}$$

$$\boldsymbol{\sigma}(x,t)\mathbf{n}da=\mathbf{P}(X,t)\mathbf{N}dA, \tag{2.174}$$

$$\boldsymbol{\sigma}(x,t)da=\mathbf{P}(X,t)dA, \tag{2.175}$$

$$\boldsymbol{\sigma}=J^{-1}\mathbf{P}\mathbf{F}^{T}, \tag{2.176}$$

$$\mathbf{P}=J\boldsymbol{\sigma}\mathbf{F}^{-T}. \tag{2.177}$$

It can be shown that $\boldsymbol{\sigma}$ is symmetric (see balance laws) and **P** is asymmetric in general and follows the rule

$$\mathbf{P}\mathbf{F}^T = \mathbf{F}\mathbf{P}^T \tag{2.178}$$

In nonlinear analysis two more definitions of stress tensors are frequently used. First there is a tensor called the Kirchhoff stress tensor $\boldsymbol{\tau}$ which differs from the Cauchy stress tensor by the volume ratio J and is defined by

$$\boldsymbol{\tau} = J\boldsymbol{\sigma} \tag{2.179}$$

Another important quantity is the second Piola-Kirchhoff stress tensor **S** which does not admit a physical interpretation in terms of surface tractions. It is defined by material coordinates and is suitable for the formulation of constitutive equations, especially for solid materials. The second Piola-Kirchhoff stress tensor can be obtained by a so called pull-back operation on the contravariant tensor field $\boldsymbol{\tau}$. So we get the relation

$$\mathbf{S} = \mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-T} \tag{2.180}$$

We can further derive a relations between **S**, $\boldsymbol{\sigma}$ and **P** like

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T} = \mathbf{F}^{-1}\mathbf{P} = \mathbf{S}^T \tag{2.181}$$

and, therefore,

$$\boldsymbol{\sigma} = J^{-1}\mathbf{F}\mathbf{S}\mathbf{F}^T \tag{2.182}$$

$$\mathbf{P} = \mathbf{F}\mathbf{S}\,. \tag{2.183}$$

2.4 Basic Postulates

There are five basic postulates in continuum mechanics:

Conservation of mass

Conservation of linear momentum

Conservation of angular momentum

Conservation of energy

The entropy inequality

2.4.1 Conservation of Mass

In non-relativistic physics mass cannot be produced or destroyed. It is assumed that during a motion there are either mass sources or mass sinks, so that the mass *m* of a body is a conserved quantity. Hence, if a particle has certain mass in the reference configuration it must stay the same during a motion [3].

$$\int_{\Omega_0} \rho_0 dV = \int_{\Omega} \rho dV \tag{2.184}$$

Where Ω_0 and Ω simply denote reference and current domains (i.e. volumes). Property ρ_0 is called the reference mass density and ρ is called the spatial mass density during a motion. The spatial mass density, also known as the density in the motion, depends on place $x \in \Omega$ and time t throughout the body. Note that, ρ_0 is time-independent and intrinsically associated with the reference configuration of the body. Hence, ρ_0 depends only on the position X chosen in configuration Ω_0. If the density does not depend on $X \in \Omega_0$, i.e. $Grad\rho_0 = 0$, the configuration is said to be homogeneous [3].

Using the relation,

$$dv = JdV = \det \mathbf{F} dV \tag{2.185}$$

the local formulation of the conservation of mass can be written as

$$\rho_0 = J\rho = \rho \det \mathbf{F} \tag{2.186}$$

We can further derive dealing with the reference density as a time independent quantity the continuity equation

$$\frac{D}{Dt}(J\rho) = \dot{\rho} + \rho div\mathbf{v} = 0 \tag{2.187}$$

There $\mathbf{v} = \dot{\mathbf{x}}$ denotes the velocity and we have used $\dot{J} = Jdiv\mathbf{v}$ [16].

2.4.2 Balance of Linear Momentum

Conservation of linear momentum requires that the time rate of change of the linear momentum (i.e., mass times velocity for all particles) of a body must balance all the forces (body plus surface) that act on the body. By definition, forces act on a deformed (current) configuration, not an undeformed (reference) configuration. Hence, the linear momentum equation is most usually derived using spatial approach, though it is often more convenient to use the referential form in pseudoelasticty. For completeness, and because of the importance of understanding interrelations between these two approaches, we drive the linear momentum balance equation both ways. In a spatial approach, we have [4].

$$\frac{d}{dt}\int_{\Omega}\rho\mathbf{v}dV = \int_{\Omega}\rho\mathbf{b}dv + \int_{\partial\Omega}T^{(n)}da \tag{2.188}$$

Where v is the velocity, and b and T are the actual body force (defined per unit mass since mass remains constant) and traction vector that act on the body in the current configuration. We desire to write our equation in the form $\int(...)dv = 0$ for all Ω, which will yield a local form. To accomplish this, the order of the time differentiation and volume integration on the left hand side must be interchanged. Because the current differential volume dv varies with time, in general, we eliminate dv in favor of JdV. Thus observe that[4]

$$\frac{d}{dt}\int_{\Omega}\rho\mathbf{v}dv = \int_{\Omega_0}\frac{d}{dt}(\rho\mathbf{v}J)dV = ... = \int_{\Omega}\rho\mathbf{a}dv \tag{2.189}$$

Wherein we used the result that, from equation 3.33

$$\int_{\Omega_0}\left(\frac{d\rho}{dt}\mathbf{v}J + \rho\mathbf{v}\frac{dJ}{dt}\right)dV = \int_{\Omega_0}\left(\frac{d\rho}{dt} + \rho tr\mathbf{L}\right)\mathbf{v}JdV = 0 \tag{2.190}$$

Because the term in (…) on the right hand side is zero by the spatial mass balance equation. Next, recognize that the second term on the right hand side of equation (2 186) can be written as a volume integral by using the definition of the Cauchy stress and the divergence theorem. That is,

$$\int_{\partial\Omega}\mathbf{T}^{(n)}da=\int_{\partial\Omega}\mathbf{nt}da=\int_{\Omega}\nabla\mathbf{t}dv\,. \tag{2.191}$$

Hence, the spatial form of the linear momentum equation becomes

$$\int_{\Omega}(\rho\mathbf{a}-\nabla\mathbf{t}-\rho\mathbf{b})dv=0 \qquad \forall \quad \Omega \tag{2.192}$$

And thereby the point wise field equation, in direct and Cartesian notation, is

$$\nabla\mathbf{t}+\rho\mathbf{b}=\rho\mathbf{a}\,, \qquad \frac{\partial t_{ij}}{\partial x_i}+\rho b_j=\rho a_j\,. \tag{2.193}$$

Again, this is the most natural and thus the most familiar form of the linear momentum equation; it is typically called the equation of motion. If the acceleration is zero (or negligible), then the equation above is called the equilibrium equation.

Similarly, balance of linear momentum in referential approach is given by

$$\frac{d}{dt}\int_{\Omega_0}\rho_0\mathbf{v}dV=\int_{\Omega_0}\rho_0\mathbf{b}dV+\int_{\partial\Omega_0}\mathbf{T}^{(N)}dA\,, \tag{2.194}$$

Where $\mathbf{T}^{(N)}$ is the traction vector defined in reference configuration in terms of the actual surface forces that act on the body in the current configuration. In contrast to the left hand side of the equation (2 186) the differentiation and integration can be interchanged directly because the reference volume dV is constant. Hence, using the definition for the first Piola-Kirchoff stress and the divergence theorem, equation (2 192) can be written as

$$\int_{\Omega_0}(\rho_0\mathbf{a}-\nabla_0\mathbf{P}-\rho_0\mathbf{b})dV=0 \qquad \forall \quad \Omega_0 \tag{2.195}$$

From which the local form of the equation is

$$\nabla_0 \mathbf{P} + \rho_0 \mathbf{b} = \rho_0 \mathbf{a}, \quad \frac{\partial P_{Aj}}{\partial X_A} + \rho_0 b_j = \rho_0 a_j \qquad \textbf{(2.196)}$$

We see therefore that the referential approach is not as "natural" as the spatial approach, but the formulation is simpler because the reference configuration does not change with time; moreover, the two relations are similar, and indeed can be obtained one from the other [4].

2.4.3 Conservation of Angular Momentum

Conservation of angular momentum requires that the time rate of change of the total momentum of the body must balance the applied moments. Hence, in referential form,

$$\frac{d}{dt}\int_{\Omega_0} (\mathbf{x} \times \rho_0 \mathbf{v}) dV = \int_{\Omega_0} (\mathbf{x} \times \rho_0 \mathbf{v}) dV + \int_{\partial\Omega_0} \left(\mathbf{x} \times \mathbf{T}^{(N)}\right) dA \qquad \textbf{(2.197)}$$

Although this is a referential form, the moment is obtained using the current position vector **x** since both force and velocity are defined in β_t. [4]Equation (2 195) can be shown to yield

$$\mathbf{F.P} = \mathbf{P}^T \mathbf{F}^T \qquad \textbf{(2.198)}$$

Considering the continuity mass equation and the local form of equilibrium we get as result of this equation the symmetry of the Cauchy stress tensor

$$\sigma = \sigma^T \qquad \textbf{(2.199)}$$

$$S = S^T \qquad \textbf{(2.200)}$$

Following the definition of the second Piola-Kirchhoff stress tensor S we get also a symmetry for this tensor.Contrary to the above the first Piola-Kirchhoff stress tensor **P** is asymmetric in general as can be seen in equation (2 196) [16].

In practice, therefore, balance of angular momentum provides little information for the formulation of a boundary value problem. Rather it provides restrictions on constitutive relations that are written in terms of $\mathbf{P}, \boldsymbol{\sigma}$ or $\mathbf{S}$; that is, it requires particular symmetries to be respected in the stress strain relations [4].

2.4.4 Conservation of Energy

Conservation of energy is also known as the first law of thermodynamics. Simply, the first law asserts that the time rate of change of the total energy of a body (kinetic plus potential being accounted for via body forces) must balance the rate at which work is done on the body (via volumetric and surface heating). Because we focus on isothermal processes throughout this work, we shall not need to invoke the energy equation. Nevertheless, for completeness the referential form of the energy equation is

$$\rho_0 \frac{d\varepsilon}{dt} = \mathbf{P}^T : \frac{d\mathbf{F}}{dt} - \nabla_0 \mathbf{q}_0 + \rho_0 g \tag{2.201}$$

Where ε is the internal energy density (defined per unit mass), ∇_0 the referential del operator, q_0 the referential heat flux vector, and g a heat addition defined per unit mass.

2.4.5 Entropy Inequality

Entropy inequality is also known as the second law of thermodynamics. The referential form of the equation (as clusius-duhem equation) is

$$-\rho_0 \left(\frac{d\Psi}{dt} + \mu \frac{dT}{dt} \right) + \mathbf{P}^T : \frac{d\mathbf{F}}{dt} - \frac{1}{T} \mathbf{q}_0 \nabla_0 T \geq 0 \tag{2.202}$$

For an isothermal process with no heat transfer, the equation above reduces to

$$-\rho_0 \frac{d\Psi}{dt} + \mathbf{P}^T : \frac{d\mathbf{F}}{dt} \geq 0 \tag{2.203}$$

Which reveals that the second law of the thermodynamics can provide important information even for isothermal, mechanical processes. It is also essential to the development of a general constitutive relation for all hyperelastic materials [4].

3 HYPERELASTIC MATERIALS

3.1 Constitutive Relations

A hyperelastic material postulates the existence of a Helmholtz free-energy function Ψ defined by unit reference volume. If Ψ only depends on the deformation gradient **F**, then it is called a strain-energy function. We will use such a function $\Psi(\mathbf{F})$ and assume it to be continuous. Further, we will restrict our considerations to homogeneous materials, i.e. every particle of the material behaves in the same manner. Using a function $\Psi(\mathbf{F})$ we can define hyperelastic material by the material model [16].

$$\mathbf{P} = \frac{\partial \Psi(\mathbf{F})}{\partial \mathbf{F}} \tag{3.1}$$

This result can be obtained by defining the inner energy D_{int} with the Claudius- Planck form of the second law of thermodynamics under the assumption of perfectly elastic material (no entropy production). We get

$$D_{\mathrm{int}} = \mathbf{P} : \dot{\mathbf{F}} - \dot{\Psi} = 0 \tag{3.2}$$

$$\dot{\Psi} = \frac{\partial \Psi}{\partial t} = \frac{\partial \Psi}{\partial \mathbf{F}} : \frac{\partial \mathbf{F}}{\partial t} = \frac{\partial \Psi}{\partial \mathbf{F}} : \dot{\mathbf{F}} \tag{3.3}$$

$$D_{\mathrm{int}} = \left(\mathbf{P} - \frac{\partial \Psi}{\partial \mathbf{F}} \right) : \dot{\mathbf{F}} = 0 \tag{3.4}$$

leading to the material model (3.1). The strain energy function has to fulfill several conditions:

- normalization condition

$$\Psi(\mathbf{I}) = 0 \tag{3.5}$$

The strain energy function has to vanish in the stress free reference configuration.

$\Psi(\mathbf{F}) \geq 0$ The strain energy function has to increase with deformation

- growth conditions

  The strain energy function has to grow to infinity if $J = \det \mathbf{F}$ approaches ∞ or 0. Physically this means that it would enquire an infinite amount of strain energy to expand the continuum body to infinite range or to compress it to a point of vanishing volume [16].

3.2 Equivalent Forms of Ψ

The strain energy function Ψ has to be objective, i.e. observer independent. This means after a translation or rotation the amount of stored energy has to be unchanged

$$\Psi(\mathbf{F}) = \Psi(\mathbf{QF}) \tag{3.6}$$

with $\mathbf{Q}$ as an orthogonal tensor having the property $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$. Setting $\mathbf{Q} = \mathbf{R}^T$ as a special choice and knowing that $\mathbf{F}$ can be decomposed in a rotational and a stretching part using the orthogonal tensor $\mathbf{R}$ and the material stretch tensor $\mathbf{U}$ or the spatial stretch tensor $\mathbf{v}$ like

$$\mathbf{F} = \mathbf{RU} = \mathbf{vR} \tag{3.7}$$

we get

$$\Psi(\mathbf{F}) = \Psi(\mathbf{R}^T\mathbf{F}) = \Psi(\mathbf{R}^T\mathbf{RU}) = \Psi(\mathbf{U}) \tag{3.8}$$

This means that is independent of the rotational part of the deformation. Knowing $\mathbf{C} = \mathbf{U}^2$ and

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) \tag{3.9}$$

we can express the strain energy in terms of $\mathbf{F}$, $\mathbf{C}$ or $\mathbf{E}$

$$\Psi(\mathbf{F}) = \Psi(\mathbf{C}) = \Psi(\mathbf{E}). \tag{3.10}$$

Manipulating some straightforward matrix algebra and

the symmetry of the right Cauchy-Green deformation tensor $\mathbf{C} = \mathbf{C}^T$ (in index form $C_{ij} = C_{ji}$) we finally get

$$\frac{\partial \Psi(\mathbf{F})}{\partial \mathbf{F}} = 2\mathbf{F}\frac{\partial \Psi(\mathbf{C})}{\partial \mathbf{C}} \tag{3.11}$$

With this relation we are able to write

$$\mathbf{P} = \frac{\partial \Psi(\mathbf{F})}{\partial \mathbf{F}} = 2\mathbf{F}\frac{\partial \Psi(\mathbf{C})}{\partial \mathbf{C}} \tag{3.12}$$

$$\mathbf{S} = \mathbf{F}^{-1}\mathbf{P} = 2\frac{\partial \Psi(\mathbf{C})}{\partial \mathbf{C}} = \frac{\partial \Psi(\mathbf{E})}{\partial \mathbf{E}} \tag{3.13}$$

$$\boldsymbol{\sigma} = J^{-1}\mathbf{P}\mathbf{F}^T = 2J^{-1}\mathbf{F}\frac{\partial \Psi(\mathbf{C})}{\partial \mathbf{C}}\mathbf{F}^T = J^{-1}\mathbf{F}\frac{\partial \Psi(\mathbf{E})}{\partial \mathbf{E}}\mathbf{F}^T \tag{3.14}$$

3.3 Isotropic Hyperelastic Materials

Isotropy is based on the physical idea that the response of the material is the same in all directions. An example of an isotropic material is rubber, which has a wide range of applications. It can be shown that the important constraint on isotropy can be expressed by

$$\Psi(\mathbf{F}) = \Psi(\mathbf{F}^*) = \Psi(\mathbf{F}\mathbf{Q}^T) \tag{3.15}$$

This means that a material is isotropic if we can show that a motion of an elastic body superimposed on a particularly translated and/or rotated reference configuration leads to the same strain energy function [16].

Knowing that we can express the strain energy function in terms of the right Cauchy-Green tensor **C** we can write:

$$\Psi(\mathbf{C}) = \Psi(\mathbf{C}^*) = \Psi(\mathbf{F}^{*T}\mathbf{F}^*) \tag{3.16}$$

$$\Psi(\mathbf{C}) = \Psi(\mathbf{Q}\mathbf{F}^T\mathbf{F}\mathbf{Q}^T) \tag{3.17}$$

$$\Psi(\mathbf{C}) = \Psi(\mathbf{Q}\mathbf{C}\mathbf{Q}^T) \tag{3.18}$$

If the last equation is fulfilled for all symmetric **C** and orthogonal **Q** then Ψ is called a scalar-valued isotropic tensor function. It can further be shown that for isotropic response the strain energy function can be expressed in terms of the left Cauchy-Green tensor $\mathbf{b} = \mathbf{F}\mathbf{F}^T$ like

$$\Psi(\mathbf{C}) = \Psi(\mathbf{b}) \tag{3.19}$$

3.4 Constitutive Equations in Terms of Invariants

It can be shown that at isotropic material behavior the scalar-valued tensor function Ψ is an invariant of **C** and, therefore, can be expressed in terms of the principal invariants I_1, I_2 and I_3 of **C** which are defined by [16].

$$I_1(\mathbf{C}) = tr(\mathbf{C}) \tag{3.20}$$

$$I_2(\mathbf{C}) = \frac{1}{2}[(tr(\mathbf{C}))^2 - tr(\mathbf{C}^2)] \tag{3.21}$$

$$I_3(\mathbf{C}) = \det(\mathbf{C}) \tag{3.22}$$

If we want to express our material model using a strain-energy function Ψ in terms of the invariants we have to derive the derivatives of the invariants with respect to the right Cauchy-Green tensor **C**. Applying the chain rule we get

$$\frac{\partial \Psi(I_1, I_2, I_3)}{\partial \mathbf{C}} = \frac{\partial \Psi}{\partial I_1}\frac{\partial I_1}{\partial \mathbf{C}} + \frac{\partial \Psi}{\partial I_2}\frac{\partial I_2}{\partial \mathbf{C}} + \frac{\partial \Psi}{\partial I_3}\frac{\partial I_3}{\partial \mathbf{C}} \tag{3.23}$$

This can be used to get the following expressions for the derivatives of the invariants of **C** with respect to **C** [16].

Where ;

$$\frac{\partial I_1(\mathbf{C})}{\partial \mathbf{C}} = \mathbf{I} \tag{3.24}$$

$$\frac{\partial I_2(\mathbf{C})}{\partial \mathbf{C}} = I_1\mathbf{I} - \mathbf{C} \tag{3.25}$$

$$\frac{\partial I_3(\mathbf{C})}{\partial \mathbf{C}} = I_3\mathbf{C}^{-1} \tag{3.26}$$

Now constitutive equations for the second Piola-Kirchhoff stress tensor $\mathbf{S}$ and the Cauchy stress tensor $\boldsymbol{\sigma}$ can be obtained using (3-13) and (3.14)

$$\mathbf{S} = 2\left[\left(\frac{\partial \Psi}{\partial I_1} + I_1\frac{\partial \Psi}{\partial I_2}\right)\mathbf{I} - \frac{\partial \Psi}{\partial I_2}\mathbf{C} + I_3\frac{\partial \Psi}{\partial I_3}\mathbf{C}^{-1}\right] \tag{3.27}$$

$$\boldsymbol{\sigma} = 2J^{-1}\mathbf{F}\left[\left(\frac{\partial \Psi}{\partial I_1} + I_1\frac{\partial \Psi}{\partial I_2}\right)\mathbf{I} - \frac{\partial \Psi}{\partial I_2}\mathbf{C} + I_3\frac{\partial \Psi}{\partial I_3}\mathbf{C}^{-1}\right]\mathbf{F}^T \tag{3.28}$$

$$\boldsymbol{\sigma} = 2J^{-1}\left[I_3\frac{\partial \Psi}{\partial I_3}\mathbf{I} + \left(\frac{\partial \Psi}{\partial I_1} + I_1\frac{\partial \Psi}{\partial I_2}\right)\mathbf{b} - \frac{\partial \Psi}{\partial I_2}\mathbf{b}^2\right] \tag{3.29}$$

Finally, we have derived constitutive equations which are suitable for a representation of the strain energy function in terms of invariants, like a Mooney-Rivlin material or the Neo-Hookean material law [16].

3.5 Incompressible Hyperelastic Materials

Numerous polymeric materials can sustain finite strains without noticeable volume changes. These materials can be considered as incompressible, meaning that only isochoric motions are possible. Incompressible materials are characterized by the incompressibility constraint

$$J = \lambda_1\lambda_2\lambda_3 = 1 \tag{3.30}$$

In order to derive constitutive equations for the common incompressible materials a Lagrange multiplier has to be introduced and so we get strain energy functions of the constrained form

$$\Psi = \Psi(\mathbf{F}) - p(J-1) \tag{3.31}$$

The scalar p serves as an indeterminate Lagrange multiplier, which can be identified as a hydrostatic pressure. Note that the scalar p may only be determined from the equilibrium equations and the boundary conditions. It represents a workless reaction to the kinematics constraint on the deformation field. [3]

Differentiating equation (3.31) with respect to the deformation gradient **F** and using $\frac{\partial J}{\partial \mathbf{F}} = J\mathbf{F}^{-T}$, we arrive at a general constitutive equation for the first Piola-Kirchoff stress tensor **P**. Hence, it will be in the form

$$\mathbf{P} = -p\mathbf{F}^{-T} + \frac{\partial \Psi(\mathbf{F})}{\partial \mathbf{F}} \tag{3.32}$$

Multiplying equation 3.32 by $\mathbf{F}^{-1}$ from the left hand side, we conclude that the second Piola-Kirchoff stress tensor **S** takes on the form

$$\mathbf{S} = -p\mathbf{F}^{-1}\mathbf{F}^{-T} + \mathbf{F}^{-1}\frac{\partial \Psi(\mathbf{F})}{\partial \mathbf{F}} = -p\mathbf{C}^{-1} + 2\frac{\partial \Psi(\mathbf{F})}{\partial \mathbf{C}} \tag{3.33}$$

However, multiplying equation with $\mathbf{F}^{T}$ from the right hand side, we conclude that the symmetric Cauchy stress tensor $\boldsymbol{\sigma}$ may be expressed as

$$\boldsymbol{\sigma} = -p\mathbf{I} + \frac{\partial \Psi(\mathbf{F})}{\partial \mathbf{F}}\mathbf{F}^{T} = -p\mathbf{I} + \mathbf{F}\left(\frac{\partial \Psi(\mathbf{F})}{\partial \mathbf{F}}\right)^{T} \tag{3.34}$$

The fundamental constitutive equations 3.32, 3.33, 3.34 are the most general forms used to define incompressible Hyperelastic materials at finite strains [3].

4 EXPERIMENTAL METHOD

4.1 In Vitro Study

For the In-vitro study, 10 specimens of Achilles' tendons distracted from 6 month old lambs as seem below with the surgical operation. Then, parts of tendon were separated into three parts from the bifurcation point.

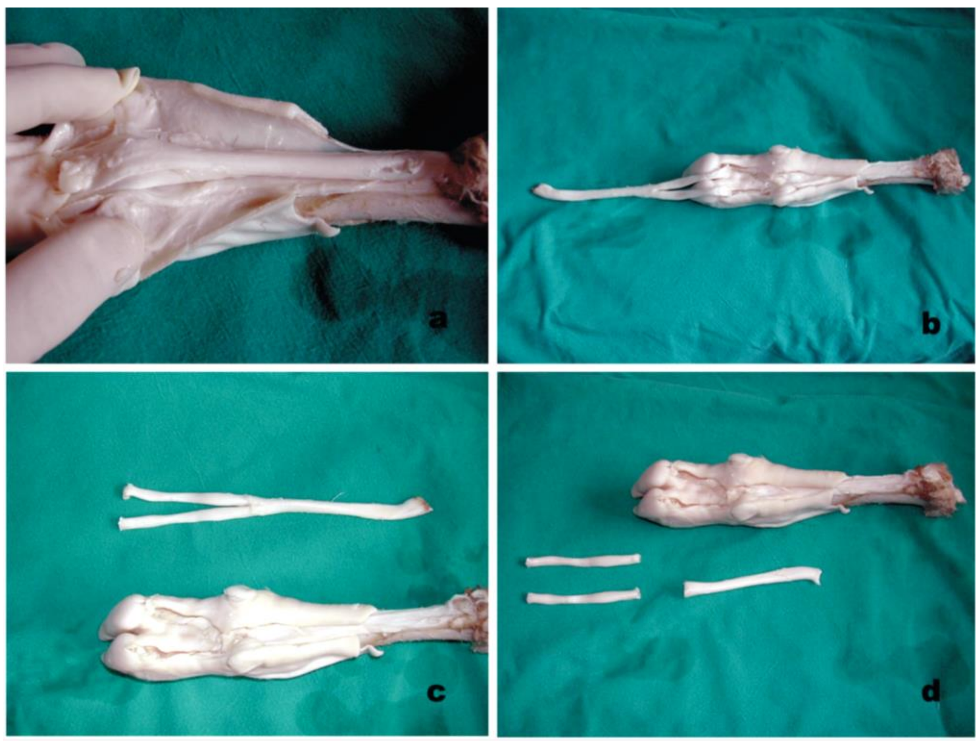


Figure 4.1: Distracting of tendon: a) Cutting of skin b) cutting of flexor tendon from the connection point c) Separating the whole tendon part from the connection point d) Separation of each tendons from the bifurcation point.

Finally, each of tendon parts were measured and 20 mm distance marked .

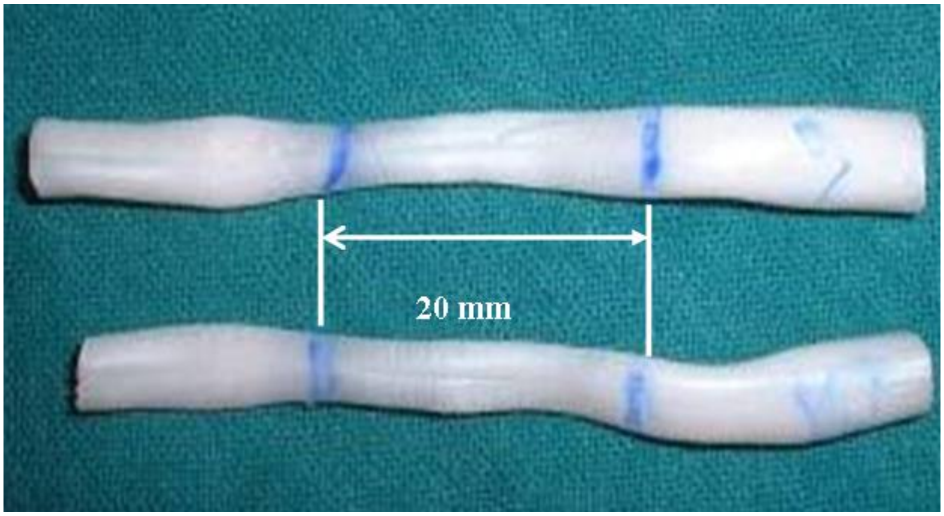


Figure 4.2: Marked specimens

Universal testing machine was used to test specimens. ESIT 100 kg capacity transducer was used for the load measurement. For the displacement measurement, VISHAY 50 mm capacity transducer was used. The load and strain data were collected with using a digital data acquisition system (Esam Traveller Plus, Computer Controlled Signal Condition Amplifier System, Esam Gmbh, Germany).

Figure 4.3: Experimental setup

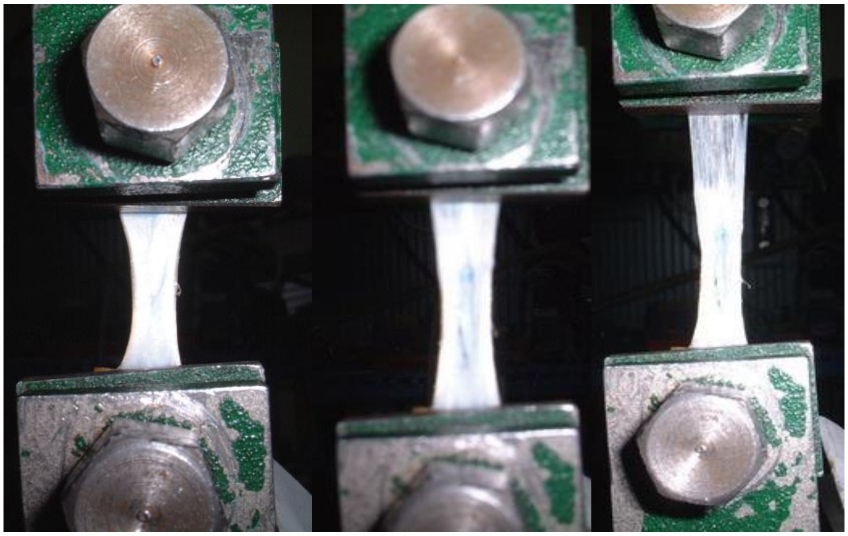


Figure 4.4: Testing of tendon specimens.

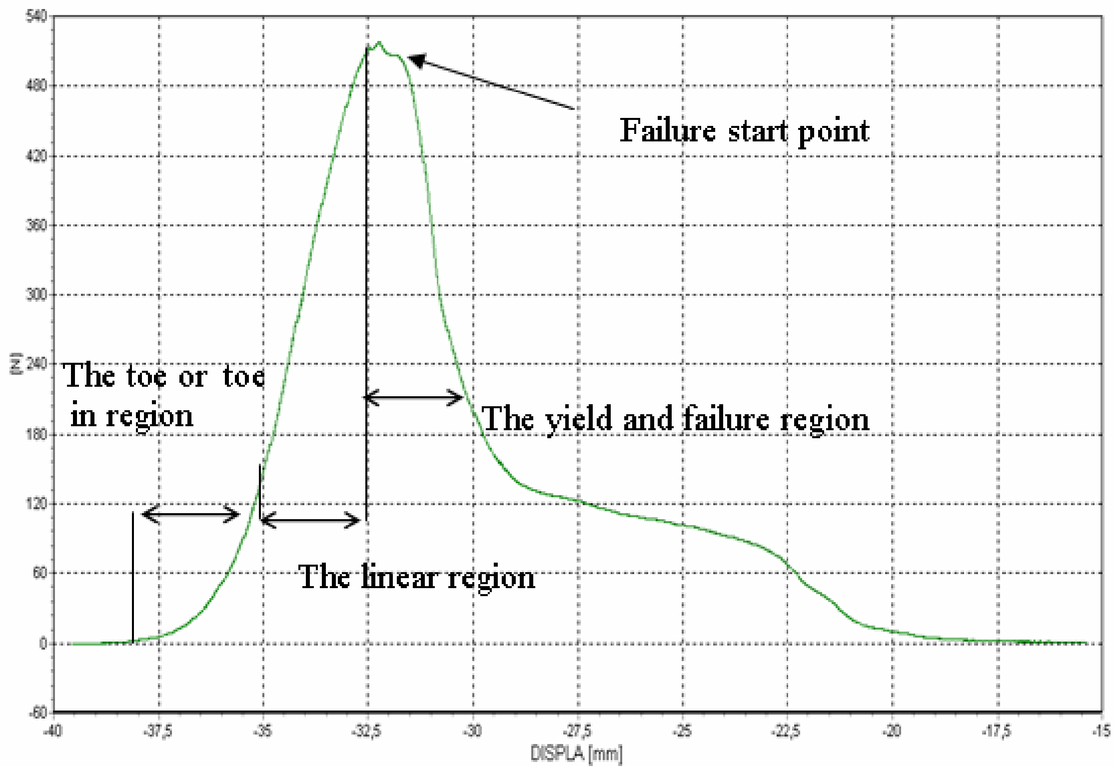


Figure 4.5: Tendon uniaxial tension graphic which gives the changes of the force against the displacement

As seen in the figure which gives the change of the force against the displacement, tendons structure behaves non linear. There are three major regions of this curve:

- the toe or toe-in region
- the linear region and
- the yield and failure region.

Most ligaments and tendons exist in the toe and somewhat in the linear region. These constitute a nonlinear stress strain curve, because of the slope of the toe-in region is different from that of the linear region.

Since it is easier to stretch out the crimp of the collagen fibrils, this part of the stress strain curve shows a relatively low stiffness. As the collagen fibrils become uncrimped, then we see that the collagen fibril backbone itself is being stretched, which gives rise to a stiffer material. As individual fibrils within the ligament or tendon begin to fail damage accumulates, stiffness is reduced and the ligament/tendons begins to fail. Since a key concept is that the overall behavior of ligaments and tendons depends on the individual crimp structure and failure of the collagen fibrils.

Hyperelastic material models such as Neo-Hookean, Monney-Rivlin, and Ogden were used to model the tendon behavior. First, these material models' constants were calculated by using the experimental results. By using the these constants and material equations, stress values were calcualted and given in the figures below. These values were compared with test data. Ogden material model with 3 parameters , found as the most appropriate for all hyper elastic models for our experimental data.

5 INCOMPRESSIBLE HYPERELASTIC MATERIAL MODELS

5.1 Definitions, Stretch ratios, Engineering Strain

To understand the hyperelastic material models, some definitions must be remembered. Some basic definitions:

Stretch ratio,

$$\lambda_i = \frac{L_i + \Delta L_i}{L_i} = 1 + \varepsilon_i \tag{5.1}$$

And engineering strain

$$\varepsilon_i = \frac{\Delta L_i}{L_i} \tag{5.2}$$

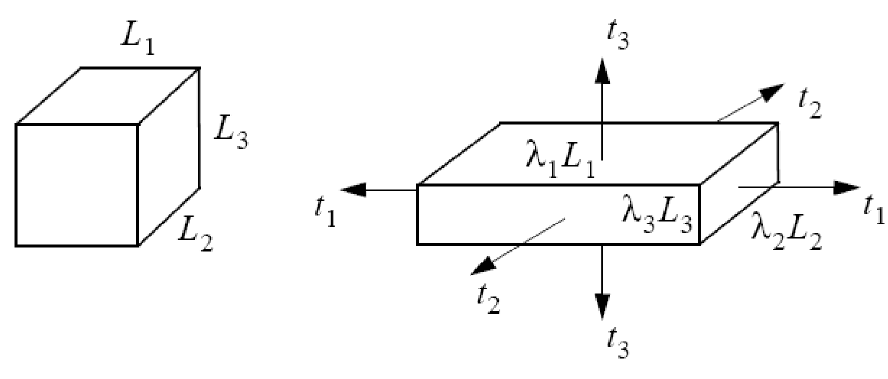


Figure 5.1: Definition of stretch ratio and engineering strain

Since the imcompressibility law**;**

$$\lambda_1 . \lambda_2 . \lambda_3 = 1 \tag{5.3}$$

5.2 Major Modes of Deformation

There major modes of deformation is noticed. They are :

- Uniaxial tension
- Biaxial Tension
- Planar shear

5.2.1 Uniaxial Tension

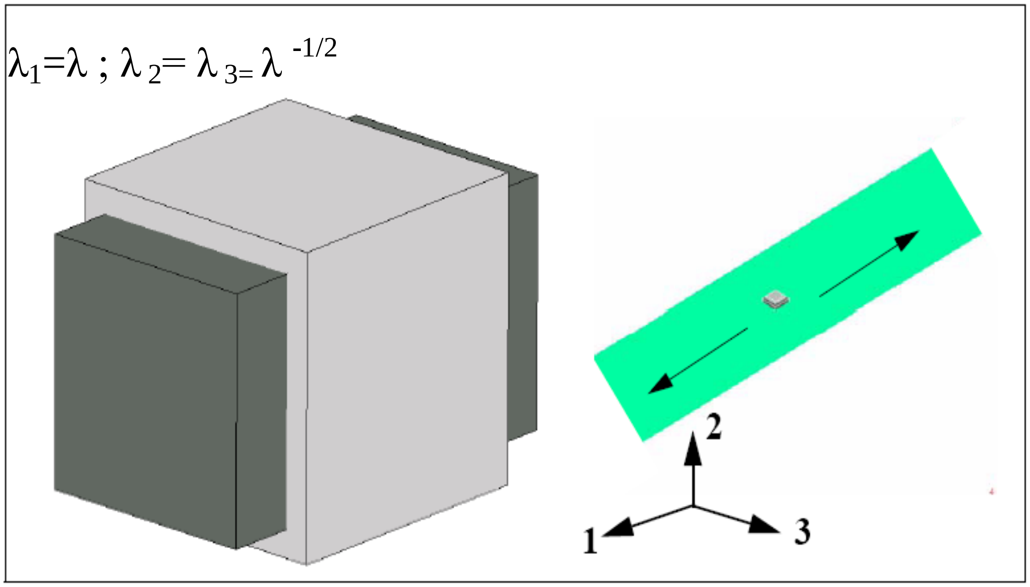


Figure 5.2: Uniaxial tension

5.2.2 Neo-Hookean Material Model For Uniaxial Tension

Firts order approximation (Neo Hookean) ;

$$W = c_{10}(\lambda_1^{\ 2} + \lambda_2^{\ 2} + \lambda_3^{\ 2} - 3) \quad \textbf{(5.4)}$$

Eliminating the λ_3 :

$$W = c_{10}(\lambda_1^{\ 2} + \lambda_2^{\ 2} + \frac{1}{\lambda_1^{\ 2}.\lambda_2^{\ 2}} - 3) \quad \textbf{(5.5)}$$

$$\mathrm{W} = c_{10}\ (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) \quad \textbf{(5.6)}$$

$$W = c_{10}\ (\lambda^2 + (1/\lambda) + (1/\lambda) - 3) \quad \textbf{(5.7)}$$

$$W = c_{10}\ (\lambda^2 + (2/\lambda) - 3) \quad \textbf{(5.8)}$$

And the stress equation ;

$$\sigma = dW/d\lambda = c_{10}\ (2\lambda - 2\lambda^{-2}) = 2\,c_{10}\ (\lambda - \lambda^{-2}) \quad \textbf{(5.9)}$$

5.2.3 Mooney-Rivlin Material Model For Uniaxial Tension:

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \quad \textbf{(5.10)}$$

$$I_2 = \lambda_1^2\ \lambda_2^2 + \lambda_2^2\ \lambda_3^2 + \lambda_1^2\ \lambda_3^2 \quad \textbf{(5.11)}$$

$$I_3 = \lambda_1 \lambda_2 \lambda_3 = 1 \quad (\Delta V = 0) \quad \textbf{(5.12)}$$

$$W = W(I_1, I_2) \quad \textbf{(5.13)}$$

$$W = C_1\ (I_1 - 3) + C_2\ (I_2 - 3) \quad \textbf{(5.14)}$$

The Mooney-Rivlin equation for the uniaxial tension ;

$$W = C_{10}\ (I_1 - 3) + C_{01}\ (I_2 - 3) \quad \textbf{(5.15)}$$

$$W = C_{10}\ (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + C_{01}\ (\lambda_1^2\ \lambda_2^2 + \lambda_2^2\ \lambda_3^2 + \lambda_1^2 \lambda_3^2 - 3) \quad \textbf{(5.16)}$$

$$W = C_{10}\ (\lambda^2 + (2/\lambda) - 3) + C_{01}\ (2\lambda + (1/\lambda)^{-2} - 3) \quad \textbf{(5.17)}$$

And the stress equation ;

$$\sigma = dW/d\lambda = 2\ C_{10}\ (\lambda - \lambda^{-2}) + 2\ C_{01}\ (1 - \lambda^{-3}) \quad \textbf{(5.18)}$$

5.2.4 Signiorini Material Model For Uniaxial Tension

$$W = C_{10}\ (I_1 - 3) + C_{01}\ (I_2 - 3) + C_{20}\ (I_1 - 3)^2 \quad \textbf{(5.19)}$$

$$W = C_{10}\ (\lambda^2 + (2/\lambda) - 3) + C_{01}\ (2\lambda + (1/\lambda)^{-2} - 3) + C_{20}\ (\lambda^2 + (2/\lambda) - 3)^2 \quad \textbf{(5.20)}$$

$C_{10}\ (I_1 - 3) + C_{01}\ (I_2 - 3)$ the derivation of this part has already taken so only ;

$d\,[\,C_{20}\,(\,\lambda^2 + (2/\lambda) - 3\,)^{\,2}\,]\,/d\lambda$

$= 2\,C_{20}\,(\,\lambda^2 + (2/\lambda) - 3\,)\;\; d\,[\,C_{20}\,(\,\lambda^2 + (2/\lambda) - 3\,)]/d\lambda$

$$=4\,C_{20}\,(\,\lambda^2 + (2/\lambda) - 3\,)\,(\,\lambda - \lambda^{-2}\,) \qquad \textbf{(5.21)}$$

$$\sigma = dW/d\lambda = 2\,C_{10}\,(\,\lambda - \lambda^{-2}\,) + 2\,C_{01}\,(\,1 - \lambda^{-3}\,) + 4\,C_{20}\,(\,\lambda^2 + (2/\lambda) - 3\,)\,(\,\lambda - \lambda^{-2}\,) \qquad \textbf{(5.22)}$$

5.2.5 Yeoh Material Model For Uniaxial Tension

$$W = C_{10}\,(\,I_1 - 3) + C_{20}\,(\,I_1 - 3)^2 + C_{30}\,(I_1 - 3)^3 \qquad \textbf{(5.23)}$$

The derivation of other parts are already taken so only;

$d\,[C_{30}\,(\,\lambda^2 + (2/\lambda) - 3\,)^3]\,/d\lambda = \;3\,C_{30}\,(\,\lambda^2 + (2/\lambda) - 3\,)^{\,2}\quad d\,[(\,\lambda^2 + (2/\lambda) - 3\,)]\,/d\lambda$

$=3\,C_{30}\,(\,\lambda^2 + (2/\lambda) - 3\,)^2 \,.\, 2C_{10}\,(\,\lambda - \lambda^{-2}\,)$

$$=6\,C_{30}\,(\,\lambda^2 + (2/\lambda) - 3\,)^{\,2}\,(\,\lambda - \lambda^{-2}\,) \qquad \textbf{(5.24)}$$

$\sigma = dW/d\lambda$

$$= 2C_{10}(\lambda - \lambda^{-2}) + 4C_{20}(\lambda^2 + (2/\lambda) - 3)(\lambda - \lambda^{-2}) + 6C_{30}(\lambda^2 + (2/\lambda) - 3)^{\,2}\,(\lambda - \lambda^{-2}\,) \qquad \textbf{(5.25)}$$

5.2.6 Ogden Model For Uniaxial Tension

The strain energy function for the Ogden model is given by:

$$W = \sum_{n=1}^{m}\left(\frac{\mu_n}{\alpha_n}\right)\left[J^{-\frac{\alpha}{3}}\left(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n}\right) - 3\right] + 4.5K\left(J^{-\frac{\alpha}{3}} - 1\right)^2 \qquad \textbf{(5.26)}$$

where U is the strain energy per unit of reference volume and μ_i ,and α_i are the material parameters. For volume constant $J=((V+\Delta V)/V) = 1$ so;

$W = \sum_{n=1}^{m}\left(\frac{\mu_n}{\alpha_n}\right)\left[\left(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n}\right) - 3\right]$. For the 3 parameter model;

$$W = \frac{\mu_1}{\alpha_1}(\lambda_1^{\alpha_1} + \lambda_2^{\alpha_1} + \lambda_3^{\alpha_1} - 3) + \frac{\mu_2}{\alpha_2}(\lambda_1^{\alpha_2} + \lambda_2^{\alpha_2} + \lambda_3^{\alpha_2} - 3) + \frac{\mu_3}{\alpha_3}(\lambda_1^{\alpha_3} + \lambda_2^{\alpha_3} + \lambda_3^{\alpha_3} - 3) \qquad \textbf{(5.27)}$$

$$\sigma = \frac{dW}{d\lambda} = \frac{(-\lambda^{-1-\frac{\alpha_1}{2}}\alpha_1 + -\lambda^{-1+\alpha_1}\alpha_1)\mu_1}{\alpha_1} + \frac{(-\lambda^{-1-\frac{\alpha_2}{2}}\alpha_2 + -\lambda^{-1+\alpha_2}\alpha_2)\mu_2}{\alpha_2} + \frac{(-\lambda^{-1-\frac{\alpha_3}{2}}\alpha_3 + -\lambda^{-1+\alpha_3}\alpha_3)\mu_3}{\alpha_3} \qquad \textbf{(5.28)}$$

5.3 Biaxial Tension (equivalent strain as uniaxial compression)

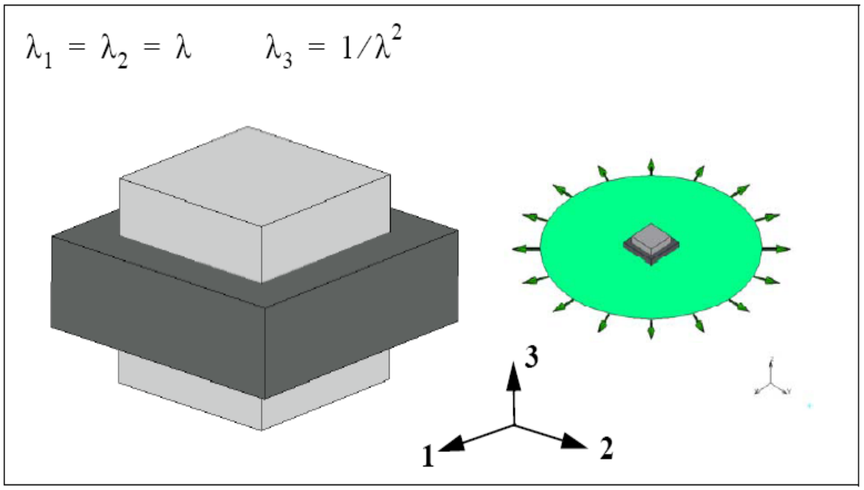


Figure 5.3: Biaxial tension

$\lambda_1.\lambda_2.\lambda_3 = 1$ and for biaxial tension $\lambda_1 = \lambda_2 = \lambda$ ve $\lambda_3 = 1/\lambda^2$

5.3.1 Neo-Hookean Model For Biaxial Tension

$$W = (G/2)(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) \qquad \textbf{(5.29)}$$

$$W = (G/2)(\lambda^2 + \lambda^2 + \lambda^{-4} - 3) \qquad \textbf{(5.30)}$$

$$\sigma = dW/d\lambda = (G/2)(4\lambda - 4\lambda^{-5}) \qquad \textbf{(5.31)}$$

$$\sigma = dW/d\lambda = 2G(\lambda - \lambda^{-5}) = 2G((1+\varepsilon) - (1+\varepsilon)^{-5}) \qquad \textbf{(5.32)}$$

5.3.2 Mooney-Rivlin Material Model For Biaxial Tension

$$W = C_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + C_2((1/\lambda_1)^2 + (1/\lambda_2)^2 + (1/\lambda_3)^2 - 3) \qquad \textbf{(5.33)}$$

$$W = C_1(\lambda^2 + \lambda^2 + \lambda^{-4} - 3) + C_2(\lambda^2 + \lambda^2 + \lambda^{-4} - 3) \qquad \textbf{(5.34)}$$

$$\sigma = dW/d\lambda = C_1(4\lambda - 4\lambda^{-5}) + C_2(-4\lambda^{-3} + 4\lambda^3) \qquad \textbf{(5.35)}$$

$$\sigma = dW/d\lambda= 4 (C_1 (\lambda - \lambda^{-5}) - C_2 (\lambda^{-3} - \lambda^3)) \quad (5.36)$$

5.3.3 Signiorini Material Model For Biaxial Tension

$$W=C_{10} (I_1 - 3) + C_{01} (I_2 – 3) + C_{20} (I_1 - 3)^2 \quad (5.37)$$

$$W= C_{10} (\lambda_1^2 +\lambda_2^2 +\lambda_3^2-3)+C_{01} (\lambda_1^2 \lambda_2^2 +\lambda_2^2 \lambda_3^2 +\lambda_1^2 \lambda_3^2- 3) +$$

$$C_{20} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)^2 \quad (5.38)$$

$$W=C_{10} (2\lambda^2 + \lambda^{-4} -3)+C_{01} (\lambda^4 + 2\lambda^{-2} -3)+ C_{20} (2\lambda^2 + \lambda^{-4} -3)^2 \quad (5.39)$$

The derivation of other parts are already taken so only,

$$d[C_{20} (2\lambda^2 + \lambda^{-4} -3)^2]/d\lambda=2 C_{20}(2\lambda^2 + \lambda^{-4} -3) d[(2\lambda^2+\lambda^{-4}-3)]$$

$$= 8C_{20} (2\lambda^2 + \lambda^{-4} -3) (\lambda - \lambda^{-5}) \quad (5.40)$$

$$dW/d\lambda= 4C_{10} (\lambda - \lambda^{-5}) + 4C_{01} (\lambda^3 - \lambda^{-3}) + 8 C_{20} (2\lambda^2 + \lambda^{-4} -3) (\lambda - \lambda^{-5}) \quad (5.41)$$

$$dW/d\lambda=4[C_{10} (\lambda - \lambda^{-5}) + C_{01} (\lambda^3 - \lambda^{-3}) + 2 C_{20} .(2\lambda^2 + \lambda^{-4} -3) (\lambda - \lambda^{-5})] \quad (5.42)$$

5.3.4 Yeoh Material Model For Biaxial Tension

$$W = C_{10} (I_1 - 3) + C_{20} (I_1 - 3)^2 + C_{30} (I_1 – 3)^3 \quad (5.43)$$

$$W = C_{10} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + C_{20} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)^2 +$$

$$C_{30} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)^3 \quad (5.44)$$

$$= C_{10} (2\lambda^2 + \lambda^{-4} -3)+ C_{20} (2\lambda^2 + \lambda^{-4} -3)^2+ C_{30} (2\lambda^2 + \lambda^{-4} -3)^3 \quad (5.45)$$

The derivation of other parts are already taken so only;

$$d[C_{30} (2\lambda^2 + \lambda^{-4} -3)^3] /d\lambda= 3C_{30} (2\lambda^2 + \lambda^{-4} -3)^2 d[C_{30} (2\lambda^2 + \lambda^{-4} -3)]/d\lambda$$

$$= 3C_{30} (2\lambda^2 + \lambda^{-4} -3)^2 (4\lambda - 4\lambda^{-5}) \quad (5.46)$$

$$\sigma = dW/d\lambda$$

$$=4C_{10}(\lambda - \lambda^{-5})+8C_{20}(2\lambda^2 +\lambda^{-4}-3) (\lambda -\lambda^{-5}) + 12 C_{30}(2\lambda^2 +\lambda^{-4} -3)^2(\lambda - \lambda^{-5}) \quad \textbf{(5.47)}$$

$$\sigma = dW/d\lambda=4 (\lambda - \lambda^{-5}) [C_{10} + 2C_{20} (2\lambda^2 + \lambda^{-4} -3) + 3C_{30}(2\lambda^2 + \lambda^{-4} -3)^2] \quad \textbf{(5.48)}$$

5.4 Planar Tension, Planar Shear, Pure Shear

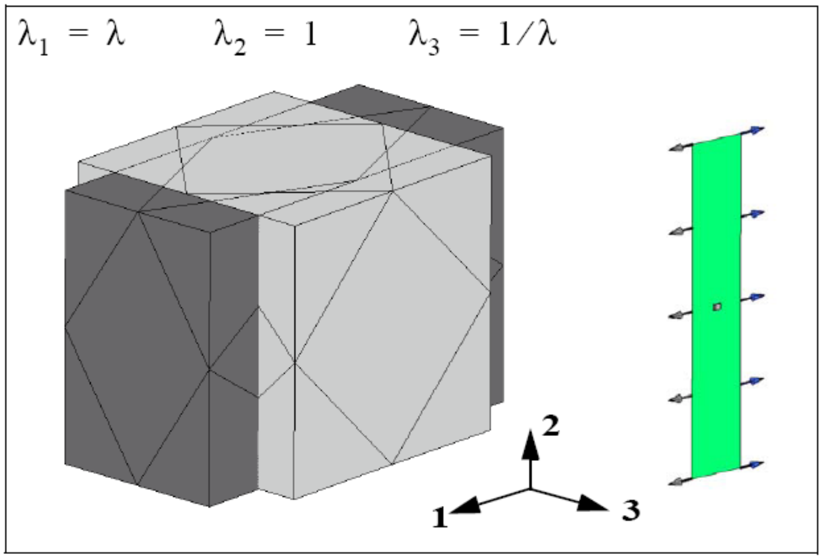


Figure 5.3: Planar tension

$\lambda_1.\lambda_2.\lambda_3 = 1$ and for the planar shear $\lambda_1=\lambda$, $\lambda_2 = 1$ ve $\lambda_3 = 1/\lambda$

5.4.1 Neo-Hookean Material Model For Planar Shear

$$W = (G/2) (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 -3) = (G/2) (\lambda^2 + \lambda^{-2}-2) \quad \textbf{(5.49)}$$

$$\sigma = dW/d\lambda=(G/2)(2\lambda-2\lambda^{-3}) = G (\lambda-\lambda^{-3}) \quad \textbf{(5.50)}$$

5.4.2 Mooney-Rivlin Material Model For Planar Shear

$$W=C_1 (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + C_2 ((1/\lambda_1)^2 + (1/\lambda_2)^2 + (1/\lambda_2)^2 - 3) \quad \textbf{(5.51)}$$

$$W=(C_1+ C_2)(\lambda^2 + \lambda^{-2}-2) \quad \textbf{(5.52)}$$

$$\sigma = dW/d\lambda = (C_1+ C_2)(2\lambda - 2\lambda^{-3}) = 2(C_1+ C_2)(\lambda - \lambda^{-3}) \quad \textbf{(5.53)}$$

5.4.3 Mooney-Rivlin Material Model For Planar Shear

$$W=C_{10}\,(I_1 - 3) + C_{01}\,(I_2 - 3) \tag{5.54}$$

$$W= C_{10}\,(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + C_{01}\,(\lambda_1^2\,\lambda_2^2 + \lambda_2^2\,\lambda_3^2 + \lambda_1^2\lambda_3^2 - 3) \tag{5.55}$$

$$W= (C_{10}+ C_{01})\,(\lambda^2 + \lambda^{-2}-2) \tag{5.56}$$

$$\sigma = dW/d\lambda= (C_{10}+ C_{01})\,(2\lambda - 2\lambda^{-3}) = 2(C_1+ C_2)(\lambda - \lambda^{-3}) \tag{5.57}$$

5.4.4 Signiorini Material Model For Planar Shear

$$W=C_{10}\,(I_1 - 3) + C_{01}\,(I_2 - 3) + C_{20}\,(I_1 - 3)^2 \tag{5.58}$$

$$W= C_{10}\,(\lambda_1^2 +\lambda_2^2 +\lambda_3^2 -3) + C_{01}(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 +\lambda_1^2\lambda_3^2 - 3) +$$

$$C_{20}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)^2 \tag{5.59}$$

$$W= 2\,(C_{10}+ C_{01})\,(\lambda^2 + \lambda^{-2}-2) + C_{20}\,(\lambda^2 + \lambda^{-2}-2)^2 \tag{5.60}$$

The derivation of other parts are already taken so only;

$$d[C_{20}\,(\lambda^2 + \lambda^{-2}-2)^2\,]/d\lambda=2(\lambda^2 + \lambda^{-2}-2)\,d[C_{20}\,(\lambda^2 + \lambda^{-2}-2)]/d\lambda$$

$$=4\,C_{20}\,(\lambda^2 + \lambda^{-2}-2)\,(\lambda - \lambda^{-3}) \tag{5.61}$$

$$\sigma = dW/d\lambda=2(C_1+ C_2)(\lambda - \lambda^{-3}) + 4C_{20}(\lambda^2 + \lambda^{-2}-2)(\lambda- \lambda^{-3}) \tag{5.62}$$

$$\sigma = dW/d\lambda=2\,(\lambda - \lambda^{-3})\,[(C_1+ C_2)+2\,C_{20}(\lambda^2 + \lambda^{-2}-2)] \tag{5.63}$$

5.4.5 Yeoh Material Model For Planar Shear

$$W=C_{10}\,(I_1 - 3) + C_{20}\,(I_1 - 3)^2 + C_{30}\,(I_1 - 3)^3 \tag{5.64}$$

$$W = C_{10}\,(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + C_{20}\,(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)^2 +$$

$$C_{30}\,(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)^3 \tag{5.65}$$

The derivation of other parts are already taken so only;

$$d[C_{30}(\lambda^2+\lambda^{-2}-2)^3]/d\lambda=6(\lambda^2+\lambda^{-2}-2)^2(\lambda-\lambda^{-3}) \quad \textbf{(5.66)}$$

$$\sigma = dW/d\lambda= 2(\lambda-\lambda^{-3})[C_{10}+2C_{20}(\lambda^2+\lambda^{-2}-2)+3C_{30}(\lambda^2+\lambda^{-2}-2)^2] \quad \textbf{(5.67)}$$

6 DATA FITTING

6.1 Theory : The Least Squares Fitting Method

The Curve fitting operations mostly uses the method of least squares when fitting data. The fitting process requires a model that relates the response data to the predictor data with one or more coefficients. The result of the fitting process is an estimate of the "true" but unknown coefficients of the model.

To obtain the coefficient estimates, the least squares method minimizes the summed square of residuals. The residual for the ith data point r_i is defined as the difference between the observed response value yi and the fitted response value $ý_i$ and is identified as the error associated with the data [5].

$$r_i = y_i + ý_i \tag{6.1}$$

residual = data - fit

The summed square of residuals is given by

$$S = \sum_{i=1}^{n} r_i^2 = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 \tag{6.2}$$

where n is the number of data points included in the fit and S is the sum of squares error estimate.The curve fitting procedures have been computed on both MSC.Patran and MATLAB environments. The theory for both software have mostly 4 types [5].

- Linear least squares
- Weighted linear least squares
- Robust least squares
- Nonlinear least squares

6.2 Linear Least Squares

A linear model is defined as an equation that is linear in the coefficients. For example, polynomials are linear but Gaussians are not. To illustrate the linear least squares fitting process, suppose you have n data points that can be modeled by a first-degree polynomial [5].

$$y = p_1 x + p_2 x \tag{6.3}$$

To solve this equation for the unknown coefficients p_1 and p_2, you write S as a system of n simultaneous linear equations in two unknowns. If n is greater than the number of unknowns, then the system of equations is overdetermined [5].

$$S = \sum_{i=1}^{n} (y_i - (p_1 x_i + p_2))^2 \tag{6.4}$$

Because the least squares fitting process minimizes the summed square of the residuals, the coefficients are determined by differentiating S with respect to each parameter, and setting the result equal to zero.

$$\frac{\partial S}{\partial p_1} = -2 \sum_{i=1}^{n} x_i (y_i - (p_1 x_i + p_2)) = 0 \tag{6.5}$$

$$\frac{\partial S}{\partial p_2} = -2 \sum_{i=1}^{n} (y_i - (p_1 x_i + p_2)) = 0 \tag{6.6}$$

The estimates of the true parameters are usually represented by b. Substituting b_1 and b_2 for p_1 and p_2, the previous equations become [5]

$$\sum x_i (y_i - (b_1 x_i + b_2)) = 0 \tag{6.7}$$

$$\sum (y_i - (b_1 x_i + b_2)) = 0 \tag{6.8}$$

where the summations run from i =1 to n. [5]

6.3 Weighted Linear Least Squares

As described in basic assumptions about the error, it is usually assumed that the response data is of equal quality and, therefore, has constant variance. If this assumption is violated, your fit might be unduly influenced by data of poor quality. To improve the fit, you can use weighted least squares regression where an additional scale factor (the weight) is included in the fitting process. Weighted least squares regression minimizes the error estimate [5].

$$S = \sum_{i=1}^{n} (w_i - (y_i + \hat{y}_i)^2 \qquad \textbf{(6.9)}$$

where w_i are the weights. The weights determine how much each response value influences the final parameter estimates. A high-quality data point influences the fit more than a low-quality data point. Weighting your data is recommended if the weights are known, or if there is justification that they follow a particular form [5].

From an engineering pont of view ; multiscale fitting problems should be conducted via "Weighted Linear Least Squares Method "; since the residual magnitudes of different orders shold be homojenized [5].

6.4 Robust Least Squares

As described in basic assumptions about the error, it is usually assumed that the response errors follow a normal distribution, and that extreme values are rare. Still, extreme values called outliers do occur. The main disadvantage of least squares fitting is its sensitivity to outliers. Outliers have a large influence on the fit because squaring the residuals magnifies the effects of these extreme data points. To minimize the influence of outliers, you can fit your data using robust least squares regression. The toolbox provides these two robust regression schemes:

Least absolute residuals (LAR) -- The LAR scheme finds a curve that minimizes the absolute difference of the residuals, rather than the squared differences. Therefore, extreme values have a lesser influence on the fit.

Bisquare weights -- This scheme minimizes a weighted sum of squares, where the weight given to each data point depends on how far the point is from the fitted line. Points near the line get full weight. Points farther from the line get reduced weight. Points that are farther from the line than would be expected by random chance get zero weight.

For most cases, the bisquare weight scheme is preferred over LAR because it simultaneously seeks to find a curve that fits the bulk of the data using the usual least squares approach, and it minimizes the effect of outliers.

Robust fitting with bisquare weights uses an iteratively re weighted least squares algorithm, and follows this procedure:

Fit the model by weighted least squares.

Compute the adjusted residuals and standardize them. The adjusted residuals are given by

$$r_{adj} = \frac{r_i}{\sqrt{1-h_i}} \qquad \textbf{(6.10)}$$

r_i are the usual least squares residuals and h_i are leverages that adjust the residuals by down weighting high-leverage data points, which have a large effect on the least squares fit. The standardized adjusted residuals are given by

$$u = \frac{r_{adj}}{Ks} \qquad \textbf{(6.11)}$$

K is a tuning constant equal to 4.685, and s is the robust variance given by MAD/0.6745 where MAD is the median absolute deviation of the residuals [5].

Compute the robust weights as a function of *u*. The bisquare weights are given by

$$w_i = \begin{cases} (1-(u_i)^2)^2 & |u_i| < 1 \\ 0 & |u_i| \geq 1 \end{cases} \qquad \textbf{(6.12)}$$

Note that if you supply your own regression weight vector, the final weight is the product of the robust weight and the regression weight.

If the fit converges, then you are done. Otherwise, perform the next iteration of the fitting procedure by returning to the first step.

The plot shown below compares a regular linear fit with a robust fit using bisquare weights. Notice that the robust fit follows the bulk of the data and is not strongly influenced by the outliers [5].

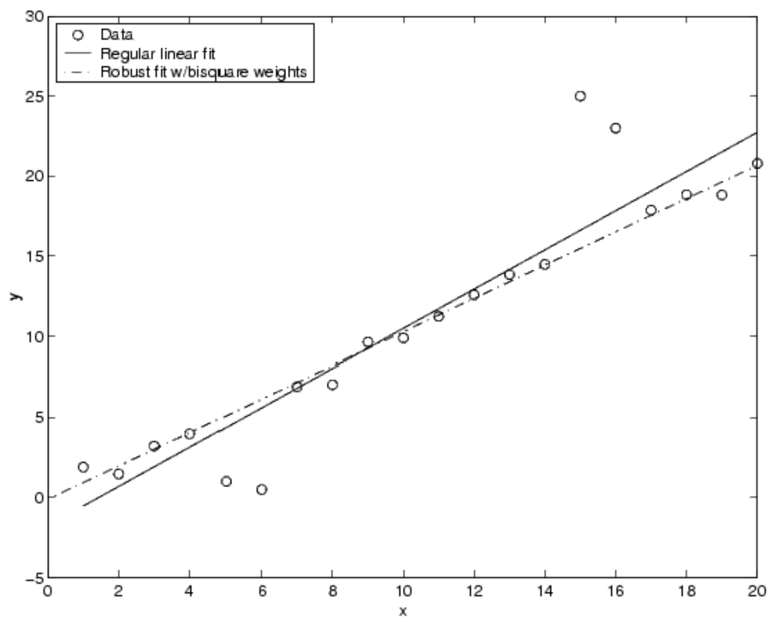


Figure 6.1: Comparising a regular linear fit with a robust fit using bisquare weights

Instead of minimizing the effects of outliers by using robust regression, you can mark data points to be excluded from the fit [5].

6.5 Nonlinear Least Squares

The nonlinear least squares formulation to fit a nonlinear model to data. A nonlinear model is defined as an equation that is nonlinear in the coefficients, or a combination of linear and nonlinear in the coefficients. For example, Gaussians, ratios of polynomials, and power functions are all nonlinear.

In matrix form, nonlinear models are given by the formula

$$y = f(X, \beta) + \varepsilon \qquad \textbf{(6.13)}$$

where

y is an n-by-1 vector of responses.

f is a function of β and X.

β is a m-by-1 vector of coefficients.

X is the n-by-m design matrix for the model.

ε is an n-by-1 vector of errors.

Nonlinear models are more difficult to fit than linear models because the coefficients cannot be estimated using simple matrix techniques. Instead, an iterative approach is required that follows these steps:

Start with an initial estimate for each coefficient. For some nonlinear models, a heuristic approach is provided that produces reasonable starting values. For other models, random values on the interval [0,1] are provided.

Produce the fitted curve for the current set of coefficients. The fitted response value is given by

$$y = f(X, \beta) + \varepsilon$$

and involves the calculation of the Jacobian of f(X,b), which is defined as a matrix of partial derivatives taken with respect to the coefficients.

Adjust the coefficients and determine whether the fit improves. The direction and magnitude of the adjustment depend on the fitting algorithm. The toolbox provides these algorithms:

- Trust-region -- This is the default algorithm and must be used if you specify coefficient constraints. It can solve difficult nonlinear problems more efficiently than the other algorithms and it represents an improvement over the popular Levenberg-Marquardt algorithm.

- Levenberg-Marquardt -- This algorithm has been used for many years and has proved to work most of the time for a wide range of nonlinear models and starting values. If the trust-region algorithm does not produce a reasonable fit, and you do not have coefficient constraints, you should try the Levenberg-Marquardt algorithm.
- Gauss-Newton -- This algorithm is potentially faster than the other algorithms, but it assumes that the residuals are close to zero [5].

6.6 Data Fitting For Incompressible Hyperelastic Material Models

6.6.1 Model Parameters

For the parameter estimation, MSC Patran software has been used. The calculated parameters for each material model showed below;

Table 6.1: Material models coefficients

Coefficients	Material Models				
	Neo Hookean	Mooney Rivlin	Signiorini	Ogden 3	Yeoh
C10	4,0116	4,0119	2,7799	0,364	37,20
C01	-	0	0	0,103	-
C11	-	-	-	0	-
C20	-	-	37,20	0	84,24
C30	-	-	-	0,162	131,06
Modulu 1	-	-	1,81	126,240	-
Modulu 2	-	-	-	0,430	-
Modulu 3	-	-	-	0	-
Exponent 1	-	-	-	0	-
Exponent 2	-	-	-	21,480	-
Exponent 3	-	-	-	6,320	-

For the parameter estimation, positive coefficients were chosen Otherwise $d\sigma. d\varepsilon<0$ and the model would have showed instable behavior. It means that; when a specimen is loaded with tension, negative stress values which is not real, may occur.

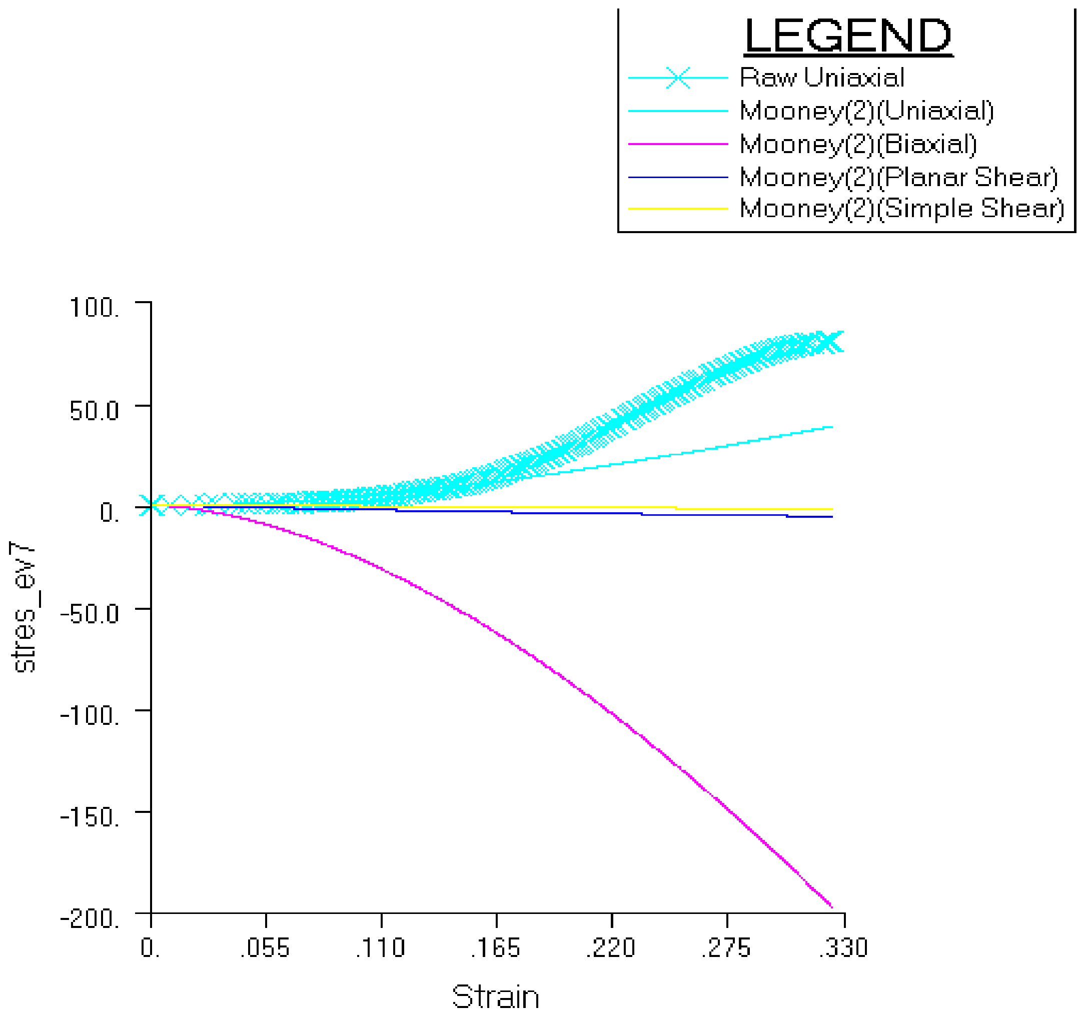


Figure 6.2: Negative coefficients result the instable behavior.

6.7 Material Model Results

6.7.1 Neo-Hookean Material

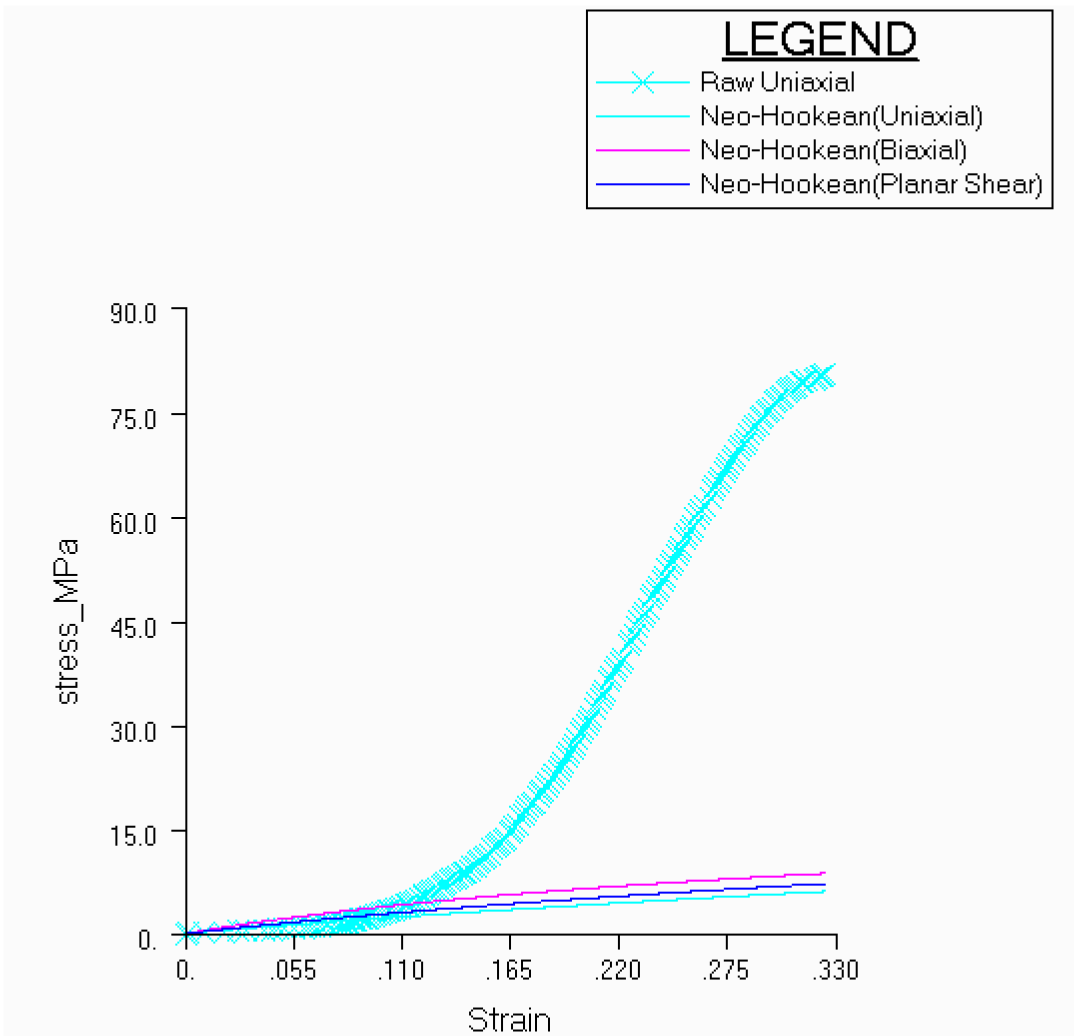


Figure 6.3: Neo Hookean model behavior

Neo-Hookean Model is the simplest nonlinear model in elongation with one coefficient.and this coefficient is not enough to stimulate whole behavior of tendon. It is asymptotic softening model of first order; thus as expected it is not suitable for fibrous soft tissues that exhibit strain-hardening at high stretches.

6.7.2 Mooney-Rivlin Material Model

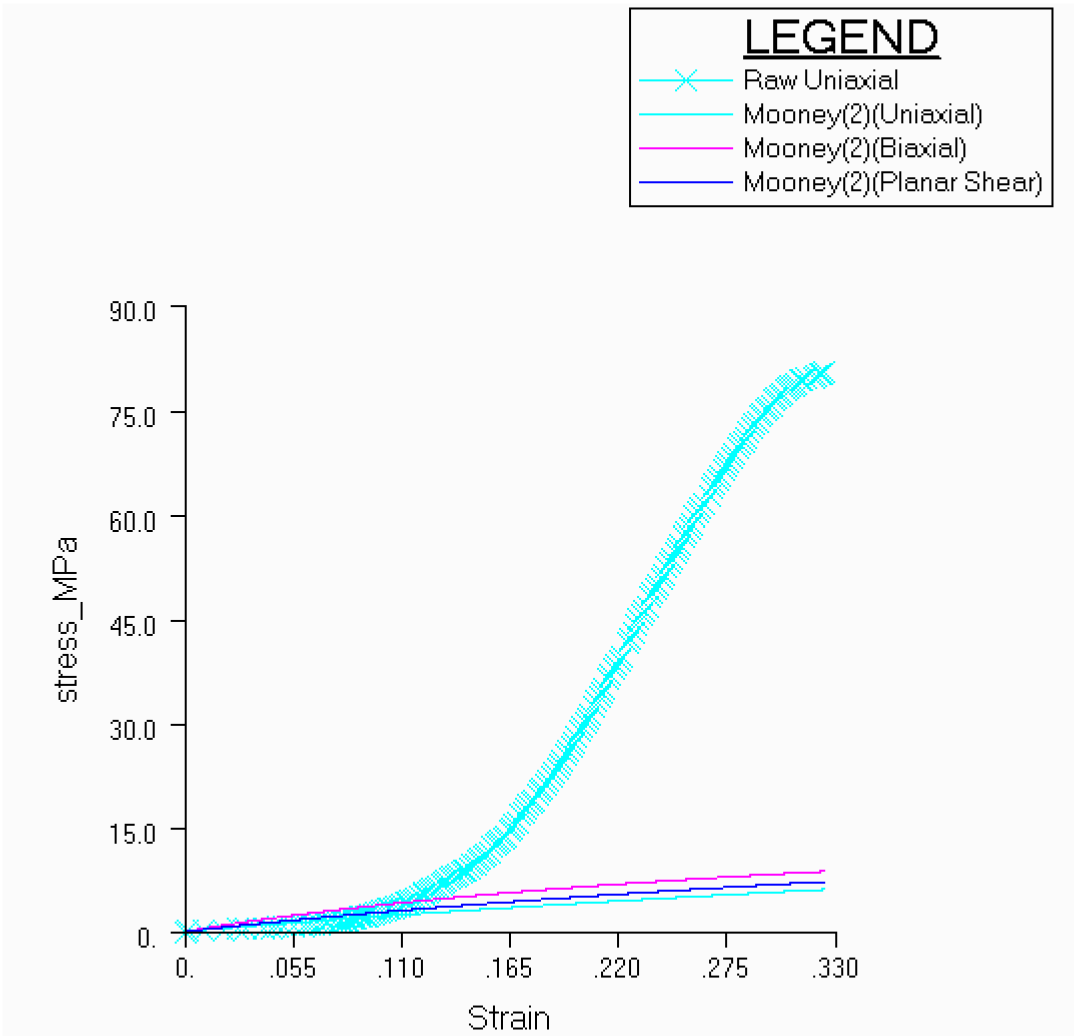


Figure 6.4: Mooney-Rivlin model behavior

Mooney-Rivlin model is bi-parametric invariant based model. During the curve fit the stability conditions imposed a “zero” coefficient for ($I_2 - 3$) term (C_{01}) Thus, Mooney-Rivlin model reduced to a Neo-Hookean model. It behaves again “strain-softening” character and is not adequate for this type of hardening materials.

6.7.3 Signiorini Material Model

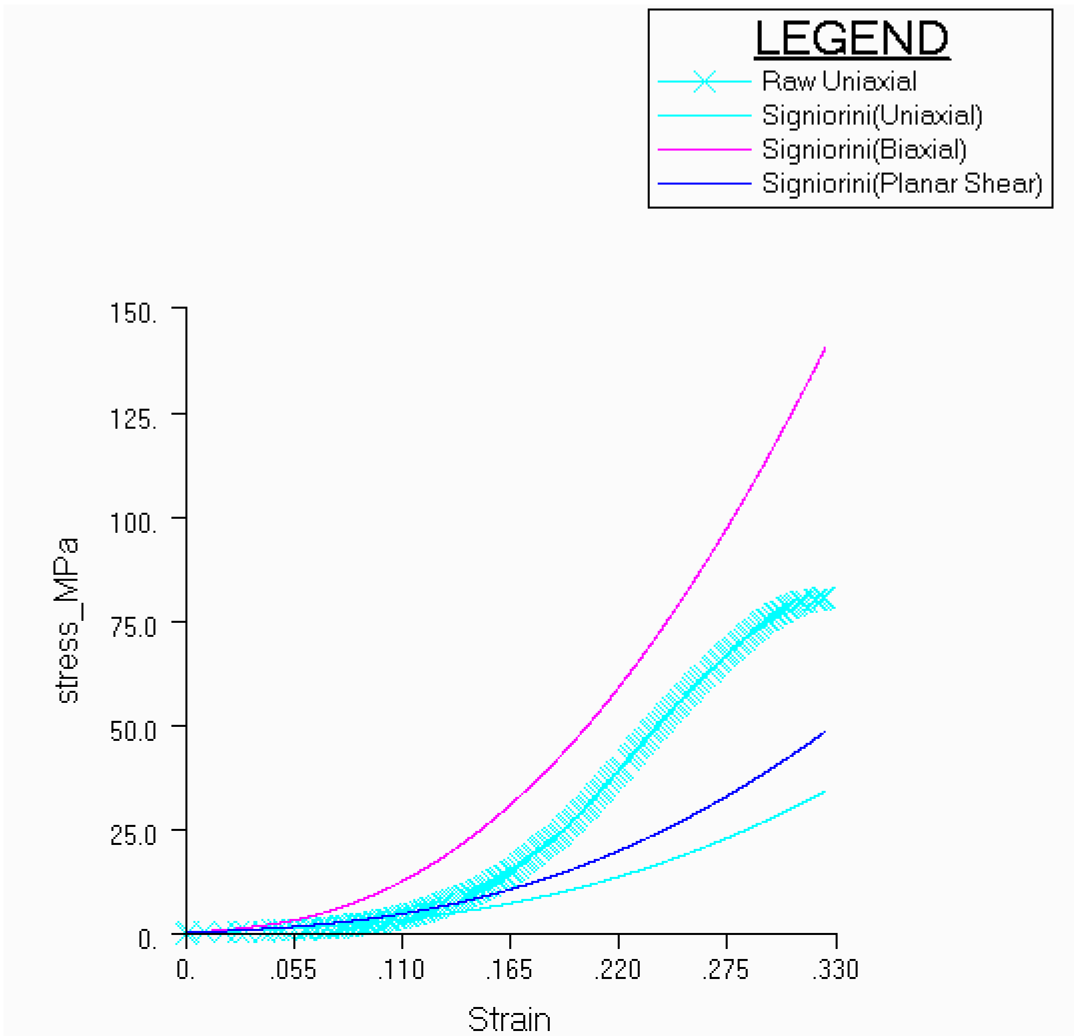


Figure 6.5: Signiorini model behavior

Considering the previous models, Signiorini model is more reasonable at lover strains (15 %). It has a stable shear and biaxial behavior. Moreover, it demonstrates stiffer biaxil characteristic. Nevertheless, it is not suitable for the high strains greater than

(15 %).

6.7.4 Yeoh Material Model

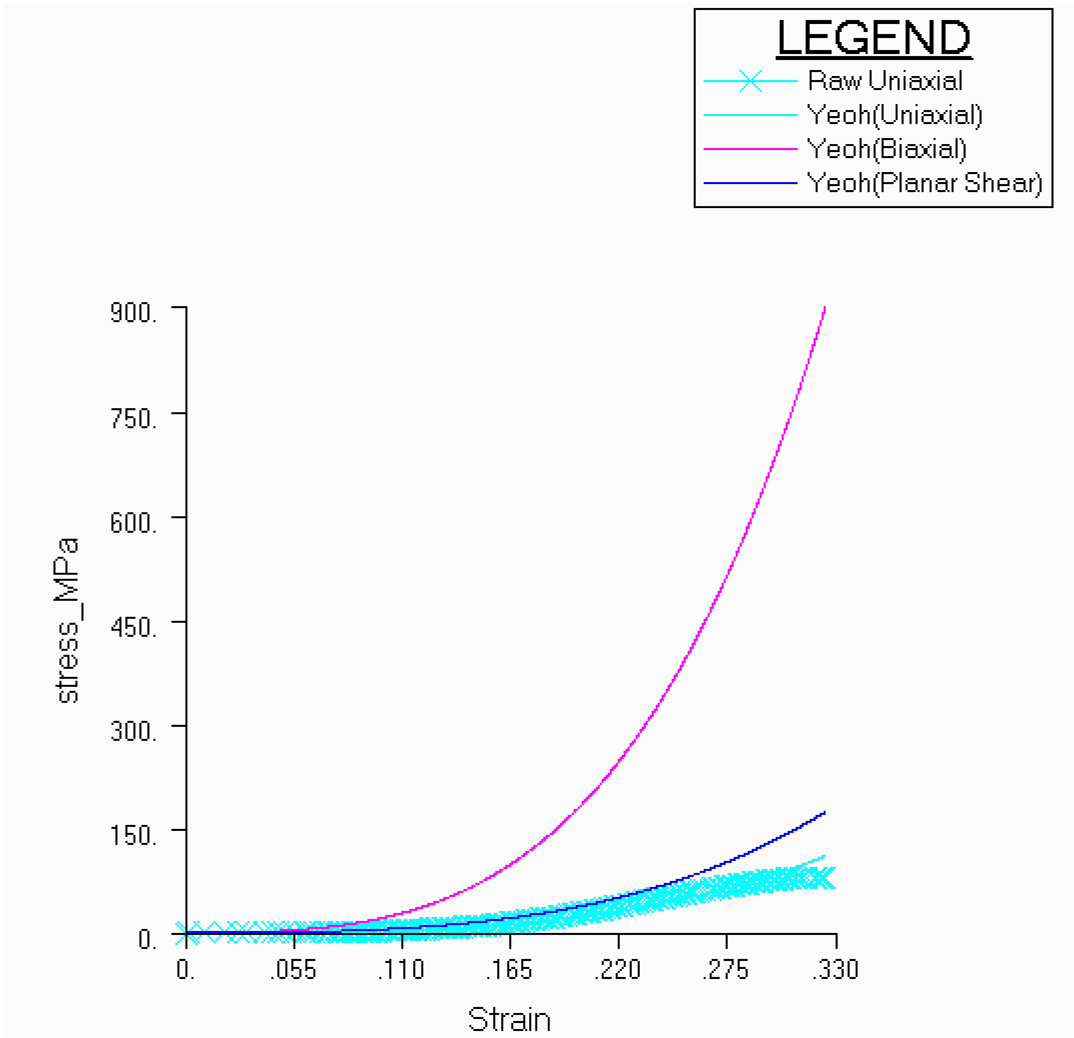


Figure 6.6: Yeoh model behavior

The Yeoh model is the best up to have. It shows reasonable behavior up to 25 % for uniaxil tension. Yeoh model seems to overestimate biaxial data. It has a third order equation with three coefficients and it makes the model more successful than the other models.

6.7.5 Ogden Model

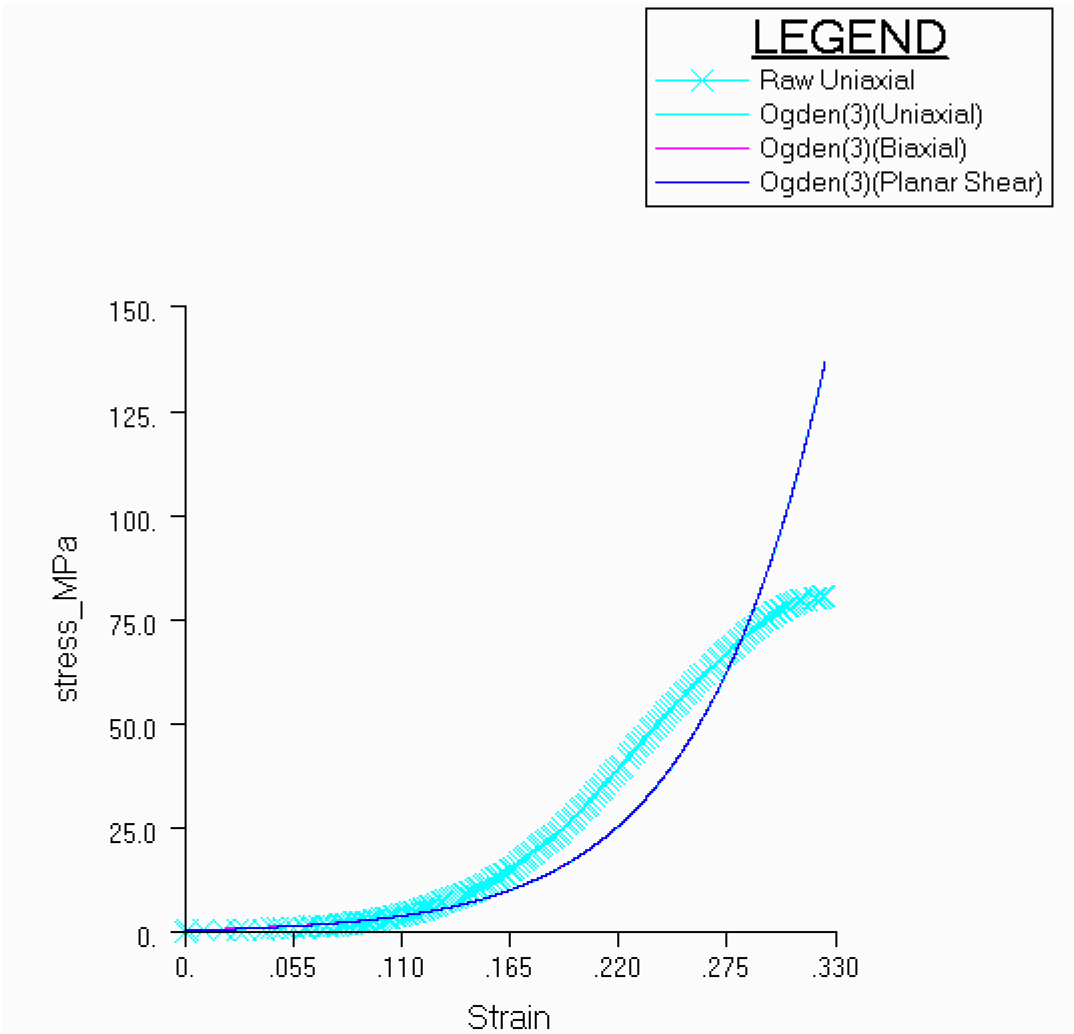


Figure 6.7: Ogden model behavior

This is the different type of Material model. Although other models are based on invariants, Ogden model is based on principle stretches for the material. Up to Ogden 2; the model does not behave stable (negative stress values under biaxial tension). Thus the most useful option is Ogden 3. Through this case; it has "all coincident" behavior under different loading conditions which seems unreasonable.

7 CONCLUSION

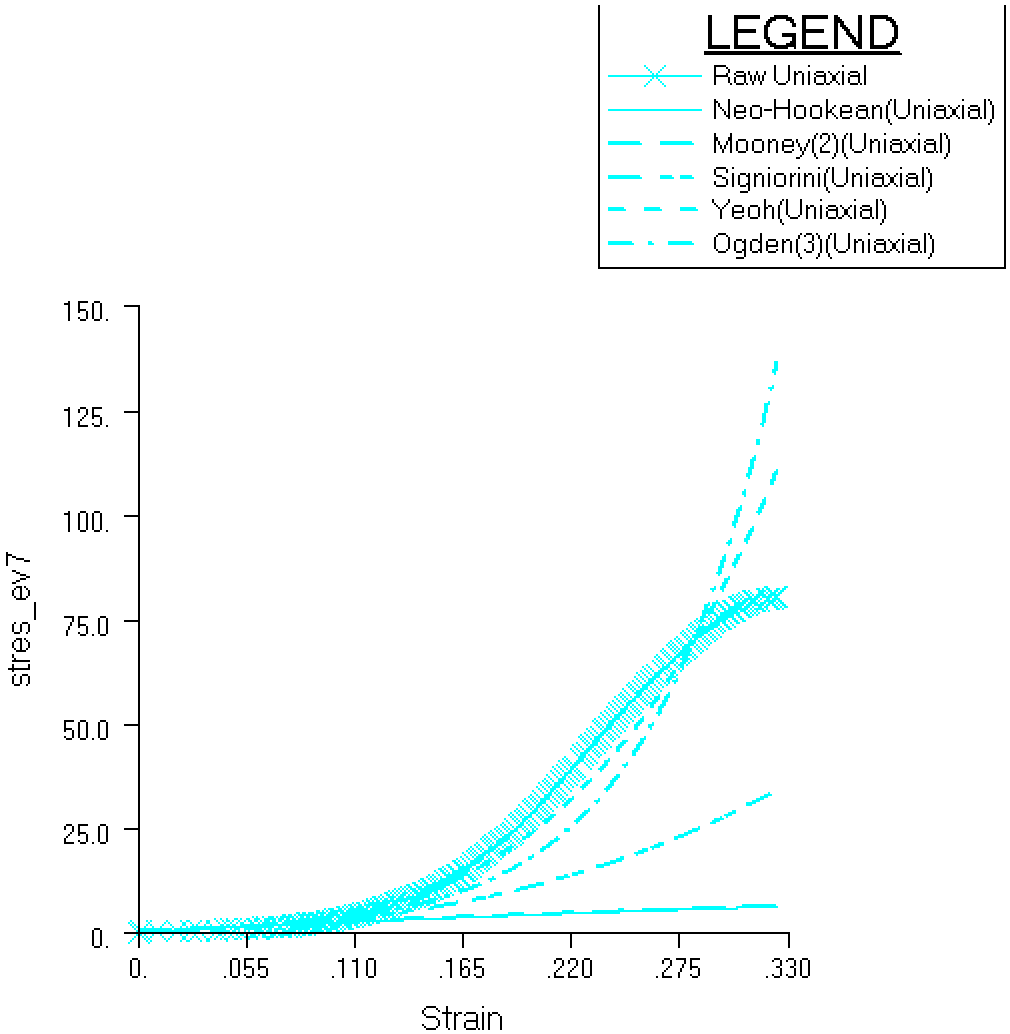


Figure 7.1: Uniaxial comparison of material models

It has been observed that the nonlinearity involved with the tendon mechanics is due to collagneous fibrous structure and its alignment mainly. Thus, the behavior is strain-hardening. This imposes that, simplest material models such as Neo-Hookean cannot be easily applied to modeling tendon behavior.

The only facilities in the laboratory, also imposed by the geometry and the structure of the tendon, have allowed uniaxial tension tests to be conducted. This restricted the variety of models that have been applied. Different modes of loading, as given to the material coefficients calculated form uniaxial tests, have been compared to simple theoretical bases, such as the stability of loading modes under different loading conditions and comparison of stiffness behavior, and no data was available to include to calculation of material properties.

It has also been observed that, the standard available material models are not sufficiently capable of representing the mechanical behavior of tendon even in uniaxial case. Thus, it is a must that material models exhibiting hardening behavior with increasing strain, unlike the Hookean model, should be applied.

The aim of such material modeling facilities is to pre-evaluate and compare different of surgical operation techniques. The calculation procedures demonstrate a great need on biaxial test on tendon structures to be exhibited, if detailed analysis of surgical operations to be simulated in computer environment.

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RESUME

I was born in 1979 in Ankara. I completed my primary and secondary education in Kent Koop İlköğretim Okulu.. Next, I started to Batıkent High School in Ankara and graduated from there in 1997.Then, I won the Mechanical Engineering Faculty of Istanbul Technical University and started my B.Sc. education in 1998. I graduated from there in 2003. Same year, I started to M.Sc. education in Solid Mechanics Program of Mechanical Engineering Department at İstanbul Technical University.

As a student of Mechanical Engineering Faculty, I have attended to Mizah student’s club and organized faculty festivals, concerts, symposiums and congress.

I am interested in biomechanics, computational mechanics and design